Instantaneously complete Ricci flows on surfaces

by

GREGOR GIESEN

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

June 2012

THE UNIVERSITY OF WARWICK

Contents

Acknowledgements

First of all, I would like to thank my supervisor Peter Topping for giving me the opportunity and support to write this thesis, for his steady guidance, constant encouragement and inspiration, as well as the fruitful collaboration throughout the past four years of my doctorate.

Many thanks to the geometric analysis group at Warwick led by Peter Topping and Mario Micallef for interesting discussions, new aspects, ideas and problems, as well as a friendly and inspiring environment; the (recent and former) members Charles Baker, Simon Blatt, Esther Cabezas-Rivas, Michael Coffey, Thomas Cuschieri, Jörg Enders, Sebastian Helmensdorfer, Vlad Moraru, Michael Munn, Huy Nguyen, Melanie Rupflin, Benjamin Sharp Fernando Schwartz, Giuseppe Tinaglia and Miaomiao Zhu and in particular the visitors Clara Aldana, Robert Haslhofer, Tobias Lamm, Rafe Mazzeo, Reto Müller, Felix Schulze, Miles Simon, Carlo Sinestrari, Michael Struwe and Burkhard Wilking.

Next, I would like to thank Rainer Nagel, Esther Cabezas-Rivas, Frank Loose and Gerhard Huisken for invitations to Tübingen in March 2010, to Münster in November 2010, to Tübingen in December 2010 and to Golm in October 2011, for their kind hospitality, stimulating discussions and new insight.

Special thanks to Peter Topping, Ben Sharp, Panagiotis Konstantis, Melanie Rupflin, Michael Coffey and Nicola Lehners for discussions, proofreading and several suggestions of improvement of this manuscript. Many thanks to my examiners, Mario Micallef and Michael Struwe, for their interesting questions and helpful comments.

Many thanks to all the new friends I made during the past four years at Warwick. These include (non-exclusively) besides the members of the geometric analysis group, Sergios Agapiou, Michael Allman, Azadeh Khajeh Alijani, Claire Kenward, Lorraine Lim, Georg Ostrovski, Cristina Sánchez-Serra, Eugenia Seah, Tanja Seel, Dayal Strub, Jorge Vitória, Miho Wako, Lee Watkins and Kathrin Wünsch.

Finally, I would like to thank my family and friends for their constant support, advice and motivation: my parents Sigrid and Heinz Giesen, my sisters Barbara Franken and Simone Giesen, my former teacher Rainer Nagel, my friends Konstantinos Blatzonis, Nicole Gebel, Panagiotis Konstantis, Reinhardt A. W. Maier, Barbara Sandhöfer and most of all Nicola Lehners.

This thesis has been supported financially by the *Research Leadership Award* of Peter Topping, funded by *The Leverhulme Trust*.

Declarations

The greater part of this thesis is a compilation of the author's joint work with Peter M. Topping which has been published in [GT10], [GT11] and [GT12]. In order to complement this survey of two-dimensional Ricci flows, in Chapter 3 we summarise known results from the 1990s about the Ricci flow on the two-sphere; here we mostly follow [BSY94] by J. Bartz, Michael Struwe and Rugang Ye. Additionally, we are giving extensive references within the text, in particular for the required background material which we provide in Chapter 2, the Appendices A and B.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

The intention of this thesis is to give a survey of *instantaneously complete* Ricci flows on surfaces, focussing on the existence and uniqueness of its Cauchy problem. We prove a general existence result for instantaneously complete Ricci flows starting at an arbitrary Riemannian surface which may be incomplete and may have unbounded curvature. We give an explicit formula for the maximal existence time, and describe the asymptotic behaviour in most cases. The issue of uniqueness within this class of instantaneously complete Ricci flows is still conjectured but we are going to describe the progress towards its proof. Finally, we apply that new existence result in order to construct an immortal complete Ricci flow which has unbounded curvature for all time.

Chapter 1

Introduction

In 1964 James Eells and Joseph H. Sampson introduced the *harmonic map heat flow* [ES64] in order to tackle geometric (and topological) problems; subsequently, the study of parabolic flows to that end has become quite popular and also very successful. Inspired by their work, in 1982 Richard S. Hamilton introduced in his seminal article [Ham82] the *Ricci flow* equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}_{g(t)} \tag{1.1}$$

and proved that for any compact three dimensional manifold $(\mathcal{M}^n, g_0)$ with positive Ricci curvature (and without boundary), there exists a unique solution $(g(t))_{t\in[0,T)}$ of (1.1) on $\mathcal{M}$ starting at $g(0) = g_0$ which converges after rescaling to a spherical space form. Consequently, he developed a program to utilise the Ricci flow in order to prove William P. Thurston's geometrisation conjecture [Thu82] about classifying three dimensional manifolds, which includes the prestigious Poincaré conjecture as a special case. After the tremendous groundwork mostly done by Hamilton, the study of Ricci flow culminated at the time, when Grigori Perelman concluded Thurston's program while introducing many innovative ideas and concepts into the theory [Per02], [Per03b], [Per03a]. Another remarkable result is the proof of the *Differential Sphere Theorem* by Simon Brendle and Richard Schoen [BS09].

1.1 Existence and uniqueness of the Cauchy problem

The initial article [Ham82] by Hamilton contains, along with the introduction of the Ricci flow, a comprehensive treatise on the existence and uniqueness of the associated Cauchy problem on compact manifolds: Given an arbitrary metric g_0 on a compact manifold $\mathcal{M}^n$ there exists a unique Ricci flow starting from g_0, either for all time or up to a final time when its curvature blows up. The situation for non-compact manifolds turns out to be much more complicated. However, Wan-Xiong Shi proved in [Shi89] a short-time existence theorem starting from a complete Riemannian manifold with bounded curvature. Yet it took another 17 years, until Bing-Long Chen and Xi-Ping Zhu provided the associated uniqueness result [CZ06], i.e. in the class of complete solutions with bounded curvature.*

In 2006, Peter M. Topping started to consider the Ricci flow on surfaces where at least one of those requirements (i.e. completeness or bounded curvature) was dropped or

*In Section 2.4 we are going to give a precise statement of these *classical* results.

weakened. Before proceeding to restrict our attention to this two dimensional situation, let us mention that recently Esther Cabezas-Rivas and Burkhard Wilking have proved new existence results on open, higher dimensional manifolds ($\dim \mathcal{M} \geq 3$) with non-negative complex sectional curvature in order to construct immortal Ricci flows with unbounded curvature [CRW11].

1.2 Ricci flows on surfaces

In the special case of *compact* surfaces, the issue of short-time existence and uniqueness of solutions of the Cauchy problem is already settled entirely by the above mentioned *classical* results. However, Hamilton [Ham88] and Bennet Chow [Cho91] rounded off the two-dimensional compact theory by describing also the Ricci flow's long-time behaviour in detail: The *area-normalised* solution of (1.1) starting at any compact Riemannian surface converges under the flow to a constant curvature metric with equal area (→ §3.1). Consequently, one obtains another proof of the *Uniformisation Theorem* A.2.4 for compact surfaces.

Then again on *non-compact* surfaces, the situation used to be quite unclear: Most existence results are either based on Shi's existence theorem or arise from the theory of the *logarithmic fast diffusion equation* (→ §4.1), but in every case they require special topological/geometrical constraints. Concerning the issue of uniqueness apart from Chen and Zhu's result only little was known until recently. In [DdP95] Panagiota Daskalapoulos and Manuel del Pino constructed a huge family of Ricci flow solutions on the plane $\mathbb{R}^2$ starting from the same initial surface. Moreover, already simple examples like the flat unit disc in the plane admit infinitely many solutions (cf. example below).

1.2.1 Instantaneously complete Ricci flows

In order to tackle this non-uniqueness phenomenon, Topping developed the concept of an *instantaneously complete* solution [Top10], that is to say even if the initial metric is incomplete, the Ricci flow solution becomes complete for any positive time.

Definition 1.2.1. A Ricci flow $\big(g(t)\big)_{t\in\mathcal{T}}$ on a surface $\mathcal{M}^2$ with the interval $\mathcal{T} \subset [0,\infty)$ including 0, is **instantaneously complete**, if $\big(\mathcal{M}, g(t)\big)$ is complete for all $t \in \mathcal{T} \setminus \{0\}$.

To justify it he proved a new very general existence result which supersedes the classical ones in two aspects: The initial surface is no longer required to be complete nor to have curvature bounded from below.

Theorem 1.2.2 (Topping [Top10, Theorem 1.1])**.** *Let $(\mathcal{M}^2, g_0)$ be a smooth Riemannian surface (without boundary) with Gaussian curvature bounded above, $\mathrm{K}_{g_0} \leq \kappa_0 \in \mathbb{R}$. Then for $T > 0$ sufficiently small so that $\kappa_0 < \frac{1}{2T}$, there exists a smooth Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on $\mathcal{M}$ such that $g(0) = g_0$ and $g(t)$ is complete for all $t \in (0,T]$. Moreover, the curvature of $g(t)$ is uniformly bounded above for all $t \in [0,T]$, and $\big(g(t)\big)_{t\in[0,T]}$ is maximally stretched*[†]*.*

Before we illustrate the particular behaviour of such instantaneously complete solutions by means of two examples, we would like to point out another non-standard way of constructing a Ricci flow on a surface: Recently Topping introduced a notion of a

[†] Roughly speaking, a Ricci flow is *maximally stretched* if it is above every competitor (→ §2.2.4).

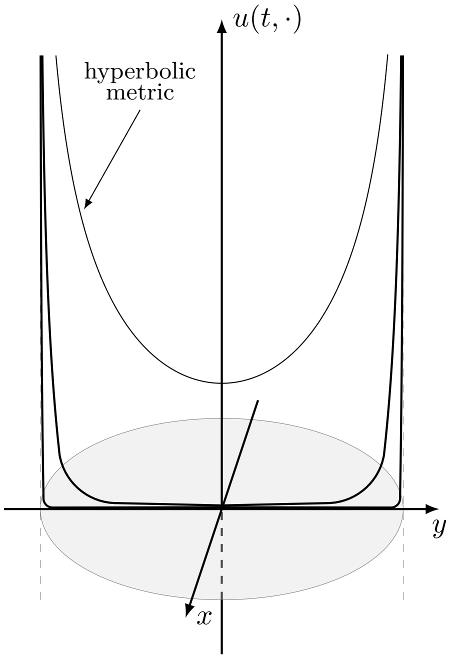


Figure 1.1: Unit disc $\mathbb{D} \subset \mathbb{R}^2$

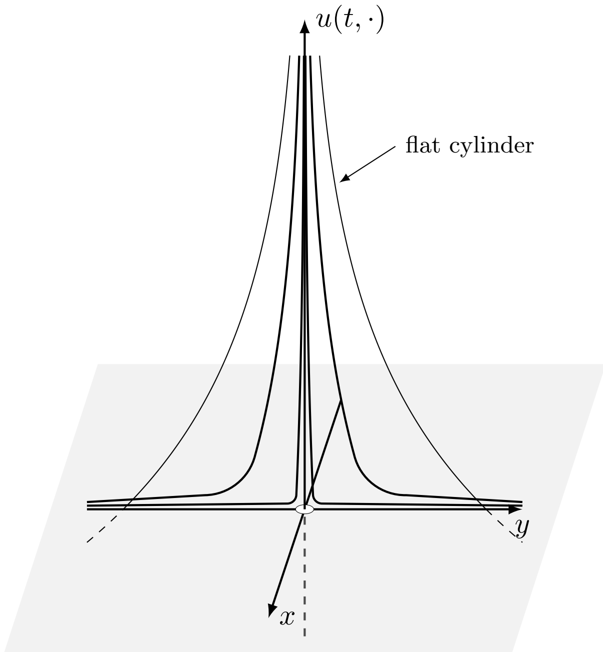


Figure 1.2: Punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$

more general initial condition which allows an 'instantaneous' change of the surface's topology in the beginning. The perfect example to think of is the complete hyperbolic cusp metric on the punctured disc as the initial surface. But instead of just flowing it using Shi's existence result, one adds a point at infinity removing the puncture and lets the flow contract the cusp as time goes on. The reason (and the prerequisite) for this non-uniqueness is the fact that the cusp is collapsed. Then again, this non-standard Ricci flow is unique among other such cusp contracting solutions. We refer to [Top12] for details and precise statements.

1.2.2 Examples

At first, we consider the flat unit disc in the plane $\mathbb{D} \subset \mathbb{R}^2$ [Top10, Example 1.5]. Since the Ricci flow equation in dimension two is conformally invariant, it can be reduced to a quasi-linear parabolic partial differential equation for the Euclidean metric's conformal factor $\mathrm{e}^{2u(t)} g_{\mathbb{E}}$ on the disc ($\rightarrow$ §2.2.1)

$$\begin{cases} \dfrac{\partial}{\partial t} u = \mathrm{e}^{-2u} \Delta u & \text{on } (0,\infty) \times \mathbb{D}, \\ \quad u \equiv 0 & \text{on } \{0\} \times \mathbb{D}. \end{cases}$$

Without prescribing Dirichlet (or Neumann) data on the boundary $\partial\mathbb{D}$ the associated Cauchy problem is ill-posed. On the other hand, by virtue of standard parabolic existence theory one could prescribe any boundary values and obtain divergent solutions, all starting from the same flat disc.

In Figure 1.1 we plot the conformal factor of the instantaneously complete solution at several times $t = 0$, $t \ll 1$, $t \approx 1$ and in the limit $t \rightarrow \infty$ (after rescaling): Obviously, the initial conformal factor $u(0) \equiv 0$ belongs to an incomplete metric. After a very short time the solution is still very close to the initial one everywhere except near the boundary, where the conformal factor blows up rapidly in order to comply with the metric's requirement to be complete. In that part it corresponds to a hyperbolic metric of very negative curvature. Later ($t \approx 1$), the negative curvature disperses towards the interior and the conformal factor starts translating upwards. It turns out that in the limit $t \to \infty$ after rescaling to compensate for the translation, the solution converges to the hyperbolic metric on $\mathbb{D}$, i.e. the complete metric of constant curvature -1. Note that Topping also showed that this particular Ricci flow starting from the flat disc is unique within the class of instantaneously complete solutions [Top10, Theorem 1.7]. This indicates that the Cauchy problem is well-posed requiring the metric to be (instantaneously) complete rather than prescribing haphazard boundary data.

The other example [Top10, Example 1.6] we are going to discuss briefly, is the flat punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ ($\to$ Figure 1.2). To overcome the incompleteness at the origin, the Ricci flow instantaneously forms a very sharp hyperbolic cusp, while away from the puncture the conformal factor stays almost constant. As time goes on, the very negative curvature of the cusp starts spreading out across the whole plane, in the same manner as on the unit disc. Though it turns out to be a unique solution ($\to$ Theorem 5.2.2), there is no *obvious* rescaling such that the flow converges to its uniformisation, the flat cylinder.

1.2.3 Main results

Since the introduction [Top10] of *instantaneously complete* Ricci flows on surfaces in 2006, Topping and the author have elaborated the theory to the following extent: In [GT11], we proved a very general existence theorem, which in contrast to Theorem 1.2.2 provides an instantaneously complete Ricci flow solution starting from *any* smooth Riemannian surface up to the maximal existence time ($\to$ §5.1). Note that this result contains the *classical*, two dimensional theory by Hamilton and Shi as a special case. We also showed that if the surface does not admit a conformal hyperbolic metric, the instantaneously complete solution is unique in its class. On the other hand, [GT10] provides a uniqueness result for a strictly negatively curved solution in that context. This restriction has been slightly relaxed since then ($\to$ §5.2). However, in [GT12] we constructed examples of complete immortal Ricci flows with unbounded curvature for all time; the above mentioned uniqueness results do not cover all these particular solutions. Finally, the article [GT11] also deals with the long-time behaviour of conformally hyperbolic solutions: The instantaneously complete Ricci flow on a surface which admits a complete conformal hyperbolic metric, uniformises (at least locally) after rescaling ($\to$ §5.3). Hence, along with the corresponding result of Hamilton and Chow for the compact cases, this complements the issue of long-time behaviour in most cases; merely on the plane does the situation remain more intricate ($\to$ §5.3.2).

1.3 Outline

This thesis consists roughly of three parts (and an appendix): This introduction and the presentation of the required background on Ricci flow in the next chapter are followed by the main part (Chapters 3–5) about existence, uniqueness and long-time behaviour of instantaneously complete Ricci flows on surfaces. The last part (Chapter 6) concludes with an application of the preceding existence theory in order to construct a somewhat exotic Ricci flow with unbounded curvature.

After this introduction we clarify in Chapter 2 our notation, summarise some basic facts about two-dimensional Ricci flow and state several results from the *classical* theory of the Ricci flow, e.g. maximum and comparison principles, existence, uniqueness and *a priori* estimates. Apart from the last part where we include some more recent curvature estimates by CHEN, the chapter's content can be regarded as standard and be skipped by a reader familiar with the subject.

The main part in the next three chapters about existence, uniqueness and long-time behaviour of instantaneously complete Ricci flows follows this strategy: Chapters 3 and 4 deal with simply connected surfaces, whose results are generalised to arbitrary surfaces in Chapter 5, where we state all the main results. Chapter 3 focusses on the only compact, simply connected surface, the 2-sphere. Though the results here can be considered to be *standard* nowadays, we include them in order to complement the theory. Chapter 4 contains the essentials of the theory of instantaneously complete Ricci flows: The two non-compact simply connected cases — on the flat plane and the hyperbolic disc. Whereas the results of the conformally flat case on the plane are mostly inherited from the theory of the *logarithmic fast diffusion equation* (→ §4.1) and transfered to our geometric situation, we present the theory of instantaneously complete Ricci flows on the hyperbolic disc in detail. In Chapter 5 we generalise those results from simply connected surfaces to those with arbitrary topology, and outline partial results and open questions.

Chapter 6 concludes with an application of the former results to construct an immortal Ricci flow which has unbounded curvature for all time. For the reader's convenience, we include in the appendices several results about Riemannian surfaces (§A) and parabolic regularity theory (§B) rather than just citing them within the text.

Chapter 2

Fundamentals

2.1 Notation and convention

Unless stated otherwise, $\mathcal{M}^n$ will be a smooth, connected, n-dimensional manifold without boundary, and $\mathcal{T} \subset \mathbb{R}$ a time interval, usually starting at (and including) 0. For a point $p \in \mathcal{M}$ we denote its (co)tangent space by $\mathrm{T}^{(*)}\mathcal{M}_p$. Furthermore, $\mathfrak{X}(\mathcal{M}) := \Gamma\big(\mathcal{M}; \mathrm{T}\mathcal{M}\big)$ are the smooth vector fields, and $\Gamma(\mathcal{M}; \mathrm{Sym}_2^{(+)}\mathrm{T}\mathcal{M})$ the (positively definite) bilinear form fields. For a Riemannian metric $g \in \Gamma(\mathcal{M}; \mathrm{Sym}_2^{+}\mathrm{T}\mathcal{M})$ we have the corresponding Levi-Cevita connection ${}^g\nabla$, the Riemannian curvature tensor Rm_g, the Ricci curvature Rc_g, the scalar curvature R_g, the Riemannian measure $\mathrm{d}\mu_g$, its lower k-dimensional variants $\mathrm{d}\mathcal{H}_g^k$, the geodesic ball $\mathcal{B}_g(p; r)$ in $p \in \mathcal{M}$ of radius $r > 0$, the volume vol_g and the arc length L_g. For any smooth (r, s) tensor field $T \in \Gamma\left(\mathcal{M}; \mathrm{T}^{(r,s)}\mathcal{M}\right) := \Gamma\left(\mathcal{M}; \bigotimes^r \mathrm{T}\mathcal{M} \otimes \bigotimes^s \mathrm{T}^*\mathcal{M}\right)$ we denote its C^k-norm by

$$\left\| T \right\|_{\mathrm{C}^k(\mathcal{M},g)} = \sum_{j=0}^{k} \sup_{\mathcal{M}} \left| ({}^g\nabla)^j T \right|_g.$$

Commonly, we use the term **Ricci flow** to denote a one-parameter family of metrics $\big(g(t)\big)_{t\in\mathcal{T}}$ on $\mathcal{M}^n$ which is a (classical) solution to

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}_{g(t)}, \tag{2.1}$$

rather than the partial differential equation itself.

Riemannian Surfaces

On a surface $\mathcal{M}^2$ we are going to use the Gaussian curvature $\mathrm{K}_g = \frac{1}{2}\mathrm{R}_g$ rather than the scalar curvature. If we pick a local isothermal complex coordinate $z = x + \mathrm{i}y$ on $\mathcal{U} \subset \mathcal{M}^2$, we can write the metric in terms of a scalar conformal factor $u \in \mathrm{C}^\infty(\mathcal{U})$

$$g = \mathrm{e}^{2u}|\mathrm{d}z|^2$$

where $|\mathrm{d}z|^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. In most of our cases, our surface will be homeomorphic to $\mathbb{C}$, such that we have only one global chart and $u \in \mathrm{C}^\infty(\mathcal{M})$ can be defined with respect to the flat metric $|\mathrm{d}z|^2$ on the whole surface. Unless stated otherwise, we will denote the Laplacian $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, the flat connection D or the norm $|\cdot|$ with respect to $|\mathrm{d}z|^2$ omitting the subscripted metric. Occasionally, we abuse notation by abbreviating

K_g by K_u or vol_g by vol_u.

Definition 2.1.1. A surface $\mathcal{M}^2$ is **finitely connected** if there exists a compact surface $\mathcal{N}^2$ and finitely many points $p_1, \ldots, p_k \in \mathcal{N}$ for some $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $\mathcal{M}$ is homeomorphic to $\mathcal{N} \setminus \{p_1, \ldots, p_k\}$. Then we define the **Euler characteristic** of $\mathcal{M}$ to be

$$\chi(\mathcal{M}) := \begin{cases} \chi(\mathcal{N}) - k & \text{if } \mathcal{M} \text{ is finitely connected,} \\ -\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\chi(\mathcal{N})$ denotes the Euler characteristic of the compact surface $\mathcal{N}$.

By virtue of the uniformisation theorem (Corollary A.2.5) every surface $(\mathcal{M}^2, g)$ admits a conformal complete metric $g_{\bar{\kappa}}$ of constant curvature $\mathrm{K}_{g_{\bar{\kappa}}} \equiv \bar{\kappa} \in \{1, 0, -1\}$, which we denote by $g_{\mathbb{S}}$, $g_{\mathbb{E}}$ or $g_{\mathbb{H}}$ respectively and call it **spherical**, **Euclidean** or **hyperbolic**.* Consequently, we can qualify a Riemannian surface $(\mathcal{M}^2, g)$ to be either **conformally spherical**, **Euclidean** or **hyperbolic** if it admits a *complete* metric with positive, zero or negative curvature respectively. We denote the corresponding simply connected Riemann surfaces by $\mathbb{S}^2$ for the 2-sphere, $\mathbb{C}$ for the complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the unit disc. Occasionally, we will write $\mathbb{D}_r \subset \mathbb{C}$ for a disc of radius $r > 0$.

2.2 Ricci flow on surfaces

On a two-dimensional manifold, the Ricci curvature is simply the Gaussian curvature times the metric $\mathrm{Rc}_g = \mathrm{K}_g \cdot g$, so the Ricci flow equation (2.1) becomes

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{K}_{g(t)} \cdot g(t). \tag{2.3}$$

Hence, the Ricci flow moves within a fixed conformal class, which allows us to simplify this coupled system of partial differential equations to a scalar equation for the metric's conformal factor.

2.2.1 Evolution of the metric's conformal factor

In order to relate a Ricci flow to a fixed conformal background metric, we need to relate their Gaussian curvatures.

Lemma 2.2.1 ([CK04, Lemma 5.3])**.** *Let $\varphi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ be a conformal map between Riemannian surfaces and define $u \in \mathrm{C}^\infty(\mathcal{M}_1)$ such that $\varphi^* g_2 = \mathrm{e}^{2u} g_1$. Then the Gaussian curvatures are related by*

$$\mathrm{K}_{g_2} \circ \varphi = \mathrm{e}^{-2u} \Big(-\Delta_{g_1} u + \mathrm{K}_{g_1} \Big). \tag{2.4}$$

Inserting $g = \mathrm{e}^{2u} g_0$ into (2.3) and using the relation (2.4) with $\varphi = \mathrm{id}_{\mathcal{M}}$, we obtain the evolution of the metric's conformal factor u under the Ricci flow as the quasi-linear scalar partial differential equation

$$\frac{\partial}{\partial t} u = -\mathrm{K}_g = \mathrm{e}^{-2u} \left(\Delta_{g_0} u - \mathrm{K}_{g_0} \right). \tag{2.5}$$

*Note that only the first and the latter metric are uniquely determined by the conformal structure, completeness and curvature, while the complete flat metric is only unique up to homotheties.

Since the Ricci flow in two dimensions preserves the conformal class, we can use the uniformisation theorem to restrict our analysis on simply connected surfaces to those three cases of either spherical, Euclidean or hyperbolic geometry which will be the contents of the next two chapters. In order to exploit these results for a Ricci flow on a surface with non-trivial topology, we are going to pull it back to its simply connected universal cover, where it is again a Ricci flow (see also Appendix A.2).

Remark 2.2.2. Let $\big(g(t)\big)_{t\in[0,T]}$ be a Ricci flow on a surface $\mathcal{M}^2$. Then there exist a universal covering $\pi:\widetilde{\mathcal{M}}\to\mathcal{M}$ and a smooth function $u\in\mathrm{C}^\infty([0,T]\times\widetilde{\mathcal{M}})$ such that $\big(\widetilde{\mathcal{M}},\pi^*g(t)\big)$ is isometrically either $\big(\mathbb{S}^2,\mathrm{e}^{2u(t)}g_{\mathbb{S}}\big)$, $\big(\mathbb{C},\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\big)$ or $\big(\mathbb{D},\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\big)$ for each $t\in[0,T]$. In particular, $u(t)$ is a *global* solution of

$$\frac{\partial}{\partial t}u=\mathrm{e}^{-2u}\left(\Delta_{g_{\mathbb{S}}}u-1\right)\qquad\text{if } \widetilde{\mathcal{M}} \text{ is compact,}\tag{2.6}$$

$$\frac{\partial}{\partial t}u=\mathrm{e}^{-2u}\Delta u\qquad\text{if } \widetilde{\mathcal{M}} \text{ is non-compact.}\tag{2.7}$$

2.2.2 Normalised Ricci flow

The following proposition describes the behaviour of a Ricci flow which is being dilated in time, and follows immediately from the scaling properties of geometric quantities.

Proposition 2.2.3. *Let $\big(g(t)\big)_{t\in(A,\Omega)}$ be a Ricci flow on a surface $\mathcal{M}^2$, $t_0\in(A,\Omega)$ and $\psi\in\mathrm{C}^\infty\big((A,\Omega),(0,\infty)\big)$. Define the change of the time parameter*

$$\begin{aligned}\bar t:(A,\Omega)\quad&\longrightarrow\quad\big(\bar A,\bar\Omega\big)\\ t\quad&\longmapsto\quad\int_{t_0}^{t}\psi(\tau)\,\mathrm{d}\tau\end{aligned}\tag{2.8}$$

with $\bar A:=\lim_{t\searrow A}\bar t(t)\in[-\infty,\infty)$ and $\bar\Omega:=\lim_{t\nearrow\Omega}\bar t(t)\in(-\infty,\infty]$, and inverse $t=\bar t^{-1}$. Then, the 1-parameter family of metrics

$$\bar g:=(\psi\cdot g)\circ t\;:\;\big(\bar A,\bar\Omega\big)\longrightarrow\Gamma\big(\mathcal{M};\mathrm{Sym}_2^+\mathrm{T}\mathcal{M}\big)\tag{2.9}$$

is a solution of

$$\frac{\partial}{\partial\bar t}\bar g(\bar t)=-\left(2\mathrm{K}_{\bar g(\bar t)}+\frac{\mathrm{d}(1/\psi)}{\mathrm{d}t}\circ t(\bar t)\right)\bar g(\bar t).\tag{2.10}$$

Hence, we may define the *normalised Ricci flow* as the following special case of (2.10).

Definition 2.2.4. A 1-parameter family of metrics $\big(\bar g(\bar t)\big)_{\bar t\in\bar{\mathcal{T}}}$ on a surface $\mathcal{M}^2$ is a **normalised Ricci flow** if it solves the equation

$$\frac{\partial}{\partial\bar t}\bar g(\bar t)=-2\big(\mathrm{K}_{\bar g(\bar t)}-\bar\kappa\big)\,\bar g(\bar t)\tag{2.11}$$

with $\bar\kappa\in\{1,0,-1\}$ according to the conformal type of $\big(\mathcal{M},\bar g(\bar t)\big)$.

In the theory of Ricci flows on compact surfaces, one usually chooses $\bar\kappa$ to be the average Gaussian curvature which by virtue of the Gauss-Bonnet Theorem A.1.1 is equal to $\frac{2\pi\,\chi(\mathcal{M})}{\mathrm{vol}_{g_0}\mathcal{M}}$. This way the normalised flow is volume preserving and converges exponentially fast to a metric of constant curvature $\frac{2\pi\,\chi(\mathcal{M})}{\mathrm{vol}_{g_0}\mathcal{M}}$.†

†See Section 3.1 and in particular Theorem 3.1.1 for details.

Then again on non-compact surfaces, the average Gaussian curvature might not be defined, so we are going to choose $\bar{\kappa} \in \{-1, 0, 1\}$ depending on the conformal type of the surface, which is determined by the uniformisation theorem (Corollary A.2.5). This way the normalised Ricci flow is expected to converge in most cases to a surface of constant Gaussian curvature $\bar{\kappa}$ ($\rightarrow$ §5.3). Note that in the compact case that choice of $\bar{\kappa}$ corresponds to the sign of the Euler characteristic.

Remark 2.2.5. In the setting of Definition 2.2.4, if we write locally $\mathrm{e}^{2\bar{u}(\bar{t})}|\mathrm{d}z|^2 = \bar{g}(\bar{t})$ for all $\bar{t} \in \bar{\mathcal{T}}$, then

$$\frac{\partial}{\partial \bar{t}}\bar{u} = \mathrm{e}^{-2\bar{u}}\Delta\bar{u} + \bar{\kappa}. \tag{2.12}$$

2.2.3 Evolution of geometric quantities

Proposition 2.2.6. *Let* $\big(g(t)\big)_{t\in\mathcal{T}}$ *and* $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in\bar{\mathcal{T}}}$ *be solutions of the unnormalised* (2.3) *and normalised* (2.11) *Ricci flow on a surface* $\mathcal{M}^2$ *respectively. For some vector fields* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(\mathcal{M})$*, we have the evolution of the associated* Levi-Cevita connection

$$\begin{aligned}\frac{\partial}{\partial t}\big\langle{}^{g}\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\big\rangle_g &= \frac{1}{2}\Big((\boldsymbol{Y}\mathrm{K}_g)\langle\boldsymbol{X}, \boldsymbol{Z}\rangle_g + (\boldsymbol{X}\mathrm{K}_g)\langle\boldsymbol{Y}, \boldsymbol{Z}\rangle_g - (\boldsymbol{Z}\mathrm{K}_g)\langle\boldsymbol{X}, \boldsymbol{Y}\rangle_g\Big)\\ \frac{\partial}{\partial \bar{t}}\big\langle{}^{\bar{g}}\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\big\rangle_{\bar{g}} &= \frac{1}{2}\Big((\boldsymbol{Y}\mathrm{K}_{\bar{g}})\langle\boldsymbol{X}, \boldsymbol{Z}\rangle_{\bar{g}} + (\boldsymbol{X}\mathrm{K}_{\bar{g}})\langle\boldsymbol{Y}, \boldsymbol{Z}\rangle_{\bar{g}} - (\boldsymbol{Z}\mathrm{K}_{\bar{g}})\langle\boldsymbol{X}, \boldsymbol{Y}\rangle_{\bar{g}}\Big),\end{aligned} \tag{2.13}$$

of the Gaussian curvature

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{K}_g &= \Delta_g\mathrm{K}_g + 2\mathrm{K}_g^2\\ \frac{\partial}{\partial \bar{t}}\mathrm{K}_{\bar{g}} &= \Delta_{\bar{g}}\mathrm{K}_{\bar{g}} + 2\mathrm{K}_{\bar{g}}\big(\mathrm{K}_{\bar{g}} - \bar{\kappa}\big),\end{aligned} \tag{2.14}$$

and of the volume form

$$\begin{aligned}\frac{\partial}{\partial t}\,\mathrm{d}\mu_g &= \mathrm{tr}_g\,\frac{\partial g}{\partial t} = -2\mathrm{K}_g\,\mathrm{d}\mu_g\\ \frac{\partial}{\partial \bar{t}}\,\mathrm{d}\mu_{\bar{g}} &= \mathrm{tr}_{\bar{g}}\,\frac{\partial \bar{g}}{\partial t} = -2\big(\mathrm{K}_{\bar{g}} - \bar{\kappa}\big)\,\mathrm{d}\mu_{\bar{g}}.\end{aligned} \tag{2.15}$$

For a proof in case of the unnormalised flow we refer to [Top06, §2.5], and the normalised variant follows similarly inserting $h = -2\left(\mathrm{K}_{\bar{g}(\bar{t})} - \bar{\kappa}\right)\bar{g}(\bar{t})$ into the deformation formulae of [Top06, §2.3.1].

2.2.4 Maximally stretched Ricci flows

In general, a Ricci flow on the universal cover of a Riemannian surface might not preserve the deck transformations (isometries) and hence it cannot be quotiented to obtain a solution on the base. For this to be true nevertheless, we need some uniqueness property, which is provided by *maximally stretched* Ricci flows.

Definition 2.2.7. A Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on a surface $\mathcal{M}^2$ with $T > 0$ is **maximally stretched**, if for any other Ricci flow $\big(\hat{g}(t)\big)_{t\in[0,\hat{T}]}$ on $\mathcal{M}$ with $\hat{g}(0) \leq g(0)$ we have

$$\hat{g}(t) \leq g(t) \qquad \text{for all } t \in \Big[0, \min\big\{T, \hat{T}\big\}\Big].$$

The definition of a *maximally stretched* Ricci flow can be viewed as a height-maximality of the conformal factor, i.e. $\mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ is maximally stretched if and only if we have $u(t) \geq v(t)$ for any other solution $\mathrm{e}^{2v(t)}|\mathrm{d}z|^2$ with $u(0) \geq v(0)$. Obviously, maximally stretched Ricci flows are unique in their class.

Remark 2.2.8. Let $\big(g(t)\big)_{t\in[0,T]}$ and $\big(\tilde{g}(t)\big)_{t\in[0,\tilde{T}]}$ be two conformally equivalent and maximally stretched Ricci flows on $\mathcal{M}^2$ with $g(0) = \tilde{g}(0)$. Then $g(t) = \tilde{g}(t)$ for all $t \in \big[0, \min\{T, \tilde{T}\}\big]$.

Therefore, as long as the uniqueness of instantaneously complete Ricci flows is still not proved, the 'maximal stretchedness' will be a convenient way to uniquely characterise a solution.

2.3 Maximum and comparison principles

2.3.1 Elementary comparison principle

In this subsection we clarify the statement and proof of two of the many variants of the standard weak maximum principle: In the first one we assume the conformal factors of two Ricci flows to be ordered on the parabolic boundary while in the second variant ($\rightarrow$ e.g. [Top11, Theorem B.1]) this requirement is replaced by the assumption that one conformal factor blows up uniformly towards the boundary.

Theorem 2.3.1 (Direct comparison principle I)**.** *Let $\mathcal{U} \Subset \mathbb{C}$ be an open, bounded domain and for some $T > 0$ let $u \in \mathrm{C}^{1,2}\big((0,T)\times\mathcal{U}\big) \cap \mathrm{C}\big([0,T]\times\overline{\mathcal{U}}\big)$ and $v \in \mathrm{C}^{1,2}\big((0,T)\times\mathcal{U}\big) \cap \mathrm{C}^{0,2}\big([0,T]\times\overline{\mathcal{U}}\big)$ both be solutions of the Ricci flow equation for the flat metric's conformal factor (2.7). If $v(0,z) \geq u(0,z)$ for all $z \in \mathcal{U}$ and $v(t,z) \geq u(t,z)$ for all $(t,z) \in [0,T]\times\partial\mathcal{U}$, then $v \geq u$ on $[0,T]\times\mathcal{U}$.*

Proof. For every $\varepsilon > 0$ consider

$$v_\varepsilon(t,z) := v\left(\frac{1}{\varepsilon}\log(\varepsilon t+1), z\right) + \frac{1}{2}\log(\varepsilon t+1) \qquad \text{for all } (t,z) \in [0,T]\times\overline{\mathcal{U}},$$

which is well-defined since $\frac{1}{\varepsilon}\log(\varepsilon t+1) \leq t$ for all $t \geq 0$. Observe that v_ε is a slight modification of v, with $v_\varepsilon(0,\cdot) = v(0,\cdot)$, and v_ε converges pointwise to v as $\varepsilon \searrow 0$, but in contrast to v it is a strict supersolution of the Ricci flow (2.7):

$$\begin{aligned}\left(\frac{\partial}{\partial t}v_\varepsilon - \mathrm{e}^{-2v_\varepsilon}\Delta v_\varepsilon\right)(t,z) &= \frac{1}{\varepsilon t+1}\left(\frac{\partial}{\partial t}v - \mathrm{e}^{-2v}\Delta v\right)\left(\frac{1}{\varepsilon}\log(\varepsilon t+1), z\right) + \frac{\varepsilon}{2(\varepsilon t+1)}\\ &= \frac{\varepsilon}{2(\varepsilon t+1)} > 0 \qquad \text{for } (t,z) \in [0,T]\times\mathcal{U}.\end{aligned} \tag{2.16}$$

We are going to prove $(v_\varepsilon - u) \geq 0$ on $[0,T]\times\overline{\mathcal{U}}$ and conclude the theorem's statement by letting $\varepsilon \searrow 0$.

Now assume that $(v_\varepsilon - u)$ becomes negative in $[0,T]\times\mathcal{U}$; hence, for every sufficiently small $\eta > 0$ define the time t_η at which $(v_\varepsilon - u)$ first becomes smaller than $-\eta$ by

$$t_\eta := \inf\Big\{t \in [0,T] : \min_{z\in\mathcal{U}}\big(v_\varepsilon - u\big)(t,z) \leq -\eta\Big\} \in (0,T).$$

Then there exists a minimum $z_\eta \in \mathcal{U}$ of $\big(v_\varepsilon - u\big)(t_\eta, \cdot\,)$ where

$$\big(v_\varepsilon - u\big)(t_\eta, z_\eta) = -\eta, \quad \Delta\big(v_\varepsilon - u\big)(t_\eta, z_\eta) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial t}\big(v_\varepsilon - u\big)(t_\eta, z_\eta) \leq 0.$$

Subtracting the Ricci flow equation for u (2.7) from (2.16) at this point (t_η, z_η), we find

$$\begin{aligned} 0 < \frac{\varepsilon}{2(\varepsilon T + 1)} &\leq \left(\frac{\partial}{\partial t} v_\varepsilon - \mathrm{e}^{-2v_\varepsilon} \Delta v_\varepsilon\right)(t_\eta, z_\eta) - \left(\frac{\partial}{\partial t} u - \mathrm{e}^{-2u}\Delta u\right)(t_\eta, z_\eta) \\ &= \left(\frac{\partial}{\partial t}\big(v_\varepsilon - u\big) - \mathrm{e}^{-2u}\Delta\big(v_\varepsilon - u\big) - \left(\mathrm{e}^{-2v_\varepsilon} - \mathrm{e}^{-2u}\right)\Delta v_\varepsilon\right)(t_\eta, z_\eta) \\ &\leq \Big(\big(1 - \mathrm{e}^{2\eta}\big)\,\mathrm{e}^{-2u}\Delta v_\varepsilon\Big)(t_\eta, z_\eta) \quad \xrightarrow{\eta \searrow 0} \quad 0 \qquad (\lightning) \end{aligned}$$

which is a contradiction. Therefore $v_\varepsilon \geq u$ on $[0,T] \times \mathcal{U}$, and the corresponding statement for v follows by letting $\varepsilon \searrow 0$. □

Corollary 2.3.2 (Direct comparison principle II [Top11, Theorem B.1])**.** *Let $\mathcal{U} \Subset \mathbb{C}$ be an open, bounded domain and for some $T > 0$ let $u \in \mathrm{C}^{1,2}\big((0,T) \times \mathcal{U}\big) \cap \mathrm{C}\big([0,T] \times \overline{\mathcal{U}}\big)$ and $v \in \mathrm{C}^{1,2}\big((0,T) \times \mathcal{U}\big) \cap \mathrm{C}\big([0,T] \times \mathcal{U}\big)$ both be solutions of the Ricci flow equation for the flat metric's conformal factor (2.7). Furthermore, suppose that there exists some function $V \in \mathrm{C}(\mathcal{U})$ with $V(z) \to \infty$ as $z \to \partial\mathcal{U}$ such that for each $t \in [0,T]$ we have $v(t,z) \geq V(z)$ for all $z \in \mathcal{U}$. If $v(0,z) \geq u(0,z)$ for all $z \in \mathcal{U}$, then $v \geq u$ on $[0,T] \times \mathcal{U}$.*

Proof. Since by hypothesis u is continuous on $[0,T] \times \overline{\mathcal{U}}$ and $v(t,\cdot) \geq V$ for each $t \in [0,T]$, where V blows up near the boundary $\partial\mathcal{U}$, there must exist a subdomain $\mathcal{U}' \Subset \mathcal{U}$ such that

$$(v - u) \geq 1 \qquad \text{throughout } [0,T] \times \big(\mathcal{U} \setminus \mathcal{U}'\big).$$

Therefore, we can apply Theorem 2.3.1 in order to compare u and v on $[0,T] \times \overline{\mathcal{U}'}$. □

2.3.2 Geometric maximum principles

In Section 4.4.2 we are going to use the following maximum principle in order to prove a geometric comparison principle for conformally hyperbolic Ricci flows on surfaces.

Theorem 2.3.3 (Variant of [CCG+08, Theorem 12.10])**.** *Let $\big(\mathcal{M}^n, \tilde{g}\big)$ be a complete Riemannian manifold with $\big|\mathrm{Rm}_{\tilde{g}}\big| \leq \kappa < \infty$, and for some $T > 0$ and $\delta > 0$ let $\big(g(t)\big)_{t\in[0,T]}$ be a family of complete metrics on $\mathcal{M}$ such that $g(t) \geq \delta\tilde{g}$ for all $t \in [0,T]$. Further let $\big(\boldsymbol{X}(t)\big)_{t\in[0,T]} \in \mathfrak{X}(\mathcal{M})$ be a smooth family of bounded vector fields on $\mathcal{M}$ with $\big|\boldsymbol{X}(t)\big|_{\tilde{g}} \leq b_1 < \infty$ for all $t \in [0,T]$. For some integrable function $f \in \mathrm{L}^1\big([0,T]\big)$ let $u \in \mathrm{C}^{1,2}\big([0,T] \times \mathcal{M}\big)$ satisfy*

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \boldsymbol{X}(t)\right) u(t,\cdot) \leq f(t)\, u(t,\cdot) \qquad \textit{wherever } u(t,\cdot) \geq 0, \tag{2.17}$$

and for some $b_2 < \infty$ and $o \in \mathcal{M}$

$$u(t,p) \leq \mathrm{e}^{b_2\left(\mathrm{dist}_{\tilde{g}}(o,p)+1\right)} \qquad \textit{for all } (t,p) \in [0,T] \times \mathcal{M}. \tag{2.18}$$

If $u(0) \leq 0$, then $u(t) \leq 0$ for all $t \in [0,T]$.

Proof. Similar to the proof of [CCG+08, Theorem 12.10] we replace the solution u by

$$\tilde{u}(t,p) := \mathrm{e}^{-\int_0^t f(\tau)\,\mathrm{d}\tau} u(t,p),$$

such that (2.17) becomes

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)} - \boldsymbol{X}(t)\right)\tilde{u}(t,\cdot) \le 0 \qquad \text{wherever } \tilde{u}(t,\cdot) \ge 0. \tag{2.17*}$$

Then the theorem's statement is a direct consequence of [CCG+08, Theorem 12.10]; alternatively, we could follow the proof of the very same theorem whilst using the function $\tilde{u}$ from above. □

PDE-ODE comparison

The following comparison principle allows us to compare the solution of a semi-linear parabolic equation with Ricci flow background to the solution of the associated ordinary differential equation.

Theorem 2.3.4 ([CCG+08, Theorem 12.14])**.** *Let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow on a manifold $\mathcal{M}^n$ with $\big|\mathrm{Rm}_{g(t)}\big| \le \kappa_0$ for some $\kappa_0 > 0$. Furthermore, let $\big(\boldsymbol{X}(t)\big)_{t\in[0,T]} \in \mathfrak{X}(\mathcal{M})$ be a smooth family of $g(t)$-bounded vector fields on $\mathcal{M}$ and $F \in \mathrm{C}\big([0,T]\times\mathbb{R}\big)$ such that $w \mapsto F(t,w)$ is locally Lipschitz for all $t \in [0,T]$. Suppose that $u \in \mathrm{C}^{1,2}\big([0,T]\times\mathcal{M}\big)$ satisfies*

$$\frac{\partial}{\partial t}u \le \Delta_{g(t)}u + \big\langle \boldsymbol{X}(t), \mathrm{d}u\big\rangle + F\big(t,u\big) \tag{2.19}$$

and $\big|u(t,p)\big| \le \mathrm{e}^{b(\mathrm{dist}_{g(t)}(o,p)+1)}$ for some point $o \in \mathcal{M}$ and constant $b < \infty$. Finally, for some $U_0 \in \mathbb{R}$ and $\tilde{T} > 0$, let $U \in \mathrm{C}^1\big([0,\tilde{T})\big)$ be the maximal solution of the ordinary differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}U(t) = F\big(t,U(t)\big) \\ U(0) = U_0. \end{cases} \tag{2.20}$$

If $u(0,p) \le U_0$ for all $p \in \mathcal{M}$, then $u(t,p) \le U(t)$ for all $(t,p) \in \big[0,\min\{\tilde{T},T\}\big] \times \mathcal{M}$.

2.4 Classical theory of the Ricci flow

Hamilton's original proof of short-time existence in the compact setting relies on quite complicated arguments including the *Nash-Moser inverse function theorem* because the evolution equation (2.1) is only weakly parabolic due to the diffeomorphism invariance of the Ricci tensor. Shortly after, Dennis DeTurck observed that this degeneracy can be broken by a slight modification of the equation; in fact, one adds the Lie derivative of a certain (time-dependent) vector field to the right-hand side of (1.1) and the now strictly parabolic equation can be solved by standard parabolic methods. Pulling back this solution by diffeomorphisms generated by those vector fields, one obtains the desired solution to the original problem. Consequently, this method has become known as the *DeTurck trick.*‡ Along with the corresponding results by Shi, Chen and Zhu

‡Note that on compact surfaces one can alternatively apply standard parabolic existence theory to the scalar partial differential equation (2.5) for the metric's conformal factor which is always non-degenerated in this compact setting.

for the non-compact but complete and bounded curvature setting, we summarise the **classical theory** in the following theorem.

Theorem 2.4.1 (Existence and uniqueness. Hamilton [Ham82]; DeTurck [DeT03]; Shi [Shi89]; Chen and Zhu [CZ06])**.** *Given a complete Riemannian manifold $\big(\mathcal{M}^n, g_0\big)$ with bounded curvature $\big|\mathrm{Rm}_{g_0}\big| \leq \kappa_0$, there exists a $T > 0$ depending only on n and κ_0, and a Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ with $g(0) = g_0$, bounded curvature and for which $\big(\mathcal{M}, g(t)\big)$ is complete for each $t \in [0, T]$. Moreover, any other complete, bounded curvature Ricci flow $\big(g_2(t)\big)_{t\in[0,T_2]}$ with $g_2(0) = g_0$ must agree with $\big(g(t)\big)_{t\in[0,T]}$ while both flows exist.*

Consequently, we can state Hamilton's long time existence result [Ham82, Theorem 14.1] in the more general setting of complete Ricci flows on either compact or non-compact manifolds.

Corollary 2.4.2 (Extension of a solution)**.** *For some $T \in (0, \infty)$ and $\kappa \in (0, \infty)$ let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow on a manifold $\mathcal{M}^n$ with bounded curvature $\big|\mathrm{Rm}_{g(t)}\big| \leq \kappa$ for all $t \in [0, T]$. Then there exist constants $\tau = \tau(\kappa, n) > 0$, $\hat\kappa = \hat\kappa(\kappa, n) < \infty$ and a smooth complete extension $\big(\hat g(t)\big)_{t\in[0,T+\tau]}$ such that $\hat g(t) = g(t)$ for all $t \in [0, T]$ and $\big|\mathrm{Rm}_{\hat g(t)}\big| \leq \hat\kappa$ for all $t \in [T, T + \tau]$.*

Moreover, if for some $\tau_2 > 0$ there exists another such complete extension $\big(\hat g_2(t)\big)_{t\in[0,T+\tau_2]}$ with bounded curvature and $g(t) = \hat g_2(t)$ for all $t \in [0, T]$, then $\hat g(t) = \hat g_2(t)$ for all $t \in \big[0, T + \min\{\tau, \tau_2\}\big]$.

Proof. By Theorem 2.4.1 there exist a constant $\tau = \tau(\kappa, n) > 0$ and a complete Ricci flow $\big(\hat g(t)\big)_{t\in[T,T+\tau]}$ starting at $\hat g(T) = g(T)$ with bounded curvature $\big|\mathrm{Rm}_{\hat g(t)}\big| \leq \hat\kappa < \infty$ for all $t \in [T, T + \tau]$. Combining both solutions we obtain the desired extension $\big(\hat g(t)\big)_{t\in[0,T+\tau]}$ by setting $\hat g(t) = g(t)$ for all $t \in [0, T)$.

If $\big(\hat g_2(t)\big)_{t\in[0,T+\tau_2]}$, for some $\tau_2 > 0$, is another complete Ricci flow with bounded curvature extending $\big(g(t)\big)_{t\in[0,T]}$, then $\hat g_2(t) = \hat g(t)$ for all $t \in \big[0, T + \min\{\tau, \tau_2\}\big]$ by Theorem 2.4.1. □

2.5 *A priori* curvature estimates

Applying the comparison principle Theorem 2.3.4 to the evolution of the Gaussian curvature under Ricci flow (2.14), we obtain curvature estimates under the condition that the Ricci flow is complete and has bounded curvature.

Proposition 2.5.1. *Let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow with bounded curvature on a surface $\mathcal{M}^2$. With $\kappa_\pm := \sup_{\mathcal{M}} \big[\pm \mathrm{K}_{g(0)}\big]_+$ we have*

$$\left.\begin{array}{ll} -\dfrac{1}{2t + \kappa_-^{-1}} & \text{if } \kappa_- > 0 \\ 0 & \text{if } \kappa_- = 0 \end{array}\right\} \leq \mathrm{K}_{g(t)} \leq \left\{\begin{array}{ll} \dfrac{1}{\kappa_+^{-1} - 2t} & \text{if } \kappa_+ > 0 \\ 0 & \text{if } \kappa_+ = 0 \end{array}\right. \tag{2.21}$$

for all $t \in [0, T]$ or $t \in [0, T] \cap \big[0, {}^1\!/_{2\kappa_+}\big)$ respectively.

Chen generalised this result by estimating the scalar curvature of a Ricci flow from below without requiring anything but the completeness of the solution. Since one does not have a maximum principle in this situation with possibly unbounded curvature, he uses an idea of Perelman [Per02, §8] and localises the problem by smoothly cutting off the scalar curvature with respect to the time-dependent distance to a fixed point such that the standard maximum principle is applicable.

Theorem 2.5.2 (Chen [Che09, Corollary 2.3(i)]). *Let $\big(g(t)\big)_{t\in[0,T]}$ be a smooth complete Ricci flow on a manifold $\mathcal{M}^n$. If $\mathrm{R}_{g(0)} \geq -\kappa$ for some $\kappa \in [0,\infty]$, then*

$$\mathrm{R}_{g(t)} \geq -\frac{n}{2t+\frac{n}{\kappa}} \qquad \text{for all } t \in [0,T]. \tag{2.22}$$

Choosing $\kappa = \infty$ in the theorem, we can easily transfer (2.22) to our situation of an instantaneously complete Ricci flow on a surface.

Corollary 2.5.3. *Let $\big(g(t)\big)_{t\in[0,T]}$ be a smooth instantaneously complete Ricci flow on a surface $\mathcal{M}^2$. Then*

$$\mathrm{K}_{g(t)} \geq -\frac{1}{2t} \qquad \text{for all } t \in (0,T]. \tag{2.23}$$

2.6 Pseudolocality

A more elaborate argument of Chen leads to the following pseudolocality-type result giving two-sided estimates on the curvature.

Theorem 2.6.1 (Chen [Che09, Proposition 3.9]). *Let $\big(g(t)\big)_{t\in[0,T]}$ be a Ricci flow on a surface $\mathcal{M}^2$. If we have for some $p \in \mathcal{M}$, $r_0 > 0$ and $v_0 > 0$*

(i) $\mathcal{B}_{g(t)}(p;r_0) \Subset \mathcal{M}$ *for all $t \in [0,T]$;*

(ii) $\left|\mathrm{K}_{g(0)}\right| \leq r_0^{-2}$ *on* $\mathcal{B}_{g(0)}(p;r_0)$*;*

(iii) $\mathrm{vol}_{g(0)}\,\mathcal{B}_{g(0)}(p;r_0) \geq v_0 r_0^2$,

then there exists a constant $C = C(v_0) > 0$ such that for all $t \in \big[0, \min\big\{T, \frac{1}{C}r_0^2\big\}\big]$

$$\left|\mathrm{K}_{g(t)}\right| \leq 2r_0^{-2} \qquad \text{on } \mathcal{B}_{g(t)}\Big(p;\frac{r_0}{2}\Big).$$

Corollary 2.6.2. *For some constants $\kappa_0 < \infty$ and $v_0 > 0$ let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow on a surface $\mathcal{M}^2$ such that $\left|\mathrm{K}_{g(0)}\right| \leq \kappa_0$ and $\mathrm{vol}_{g(0)}\,\mathcal{B}_{g(0)}(p;1) \geq v_0$ for all $p \in \mathcal{M}$. Then there exists a constant $C = C(v_0,\kappa_0) < \infty$ such that*

$$\left|\mathrm{K}_{g(t)}\right| \leq C \qquad \text{for all } t \in \big[0, \min\big\{T, \kappa_0^{-1}\big\}\big].$$

Chapter 3

Ricci flow on the 2-sphere

This and the next chapter are about (instantaneously complete) Ricci flows on simply connected surfaces, which can be classified by virtue of the uniformisation theorem to be conformally equivalent either to the compact sphere (this chapter), the non-compact flat plane or the non-compact hyperbolic disc (Chapter 4). Since the compact sphere $\mathbb{S}^2$ does not admit any incomplete metric, the theory of instantaneously complete Ricci flows on compact surfaces coincides with the classical one.

3.1 Ricci flow on compact surfaces

The following theorem summarises the theory of *normalised* Ricci flows on compact surfaces.

Theorem 3.1.1 (Hamilton [Ham88]; Chow [Cho91])**.** *Let $\big(\mathcal{M}^2, g_0\big)$ be a smooth, compact Riemannian surface without boundary. Then there exists a unique solution $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\infty)}$ of the normalised Ricci flow*

$$\begin{cases} \dfrac{\partial}{\partial \bar{t}}\,\bar{g}(\bar{t}) = -2\Big(\mathrm{K}_{\bar{g}(\bar{t})} - \bar{\kappa}\Big)\bar{g}(\bar{t}) \\ \quad \bar{g}(0) = g_0. \end{cases} \tag{3.1}$$

where $\bar{\kappa} = \fint_{\mathcal{M}} \mathrm{K}_{g_0}\,\mathrm{d}\mu_{g_0}$ is the average Gaussian curvature. Moreover, let g_∞ be the conformally equivalent (to g_0) metric of constant curvature $\mathrm{K}_{g_\infty} \equiv \bar{\kappa}$. Then $\bar{g}(\bar{t})$ converges smoothly and exponentially fast to g_∞ as $\bar{t} \to \infty$, i.e. for each $k \in \mathbb{N}_0$ there are constants $c = c(g_0) > 0$ and $C = C(k, g_0) < \infty$ such that

$$\Big\|\bar{g}(\bar{t}) - g_\infty\Big\|_{\mathrm{C}^k(\mathcal{M},g_0)} \leq C\,\mathrm{e}^{-c\bar{t}} \qquad \text{for all } \bar{t} \in [0,\infty). \tag{3.2}$$

Whilst Hamilton [Ham82] showed short-time existence and uniqueness for compact manifolds (of arbitrary dimension) (→ §2.4), in [Ham88, §4] he used the evolution of the *potential of curvature* (→ §3.4.2) and derived quantities in order to obtain (time-dependent) curvature bounds for all time and hence long-time existence (→ Corollary 2.4.2). In the cases of non-positive Euler characteristic $\chi(\mathcal{M}) \leq 0$, the curvature bounds are also uniform in time, and if $\chi(\mathcal{M}) < 0$, we already have exponential convergence. The case of $\chi(\mathcal{M}) = 0$ requires a slightly more detailed analysis (see [Ham88, §5] or [Gie07, §5.3] for an alternative approach without using the uniformisation theorem).

However, the case of positive Euler characteristic $\chi(\mathcal{M}) > 0$ turns out to be much more involved. In fact, if additionally the initial curvature is non-negative, Hamilton developed in [Ham88] an *entropy estimate* to prove uniform bounds for the Gaussian curvature, and a *differential Harnack inequality* to gain (after some time) a strictly positive lower bound to the curvature, which is necessary to show convergence to a shrinking gradient Ricci soliton (cf. the proofs of Proposition 3.4.4 and Theorem 3.4.5 below). But the only shrinking gradient Ricci soliton in two dimensions is a spherical space form, concluding the proof of Theorem 3.1.1 provided that the curvature of the initial surface was non-negative.

Then again in case of arbitrary initial curvature, Chow [Cho91] generalised Hamilton's entropy and differential Harnack inequality to this setting and additionally proved a lower bound for the injectivity radius, which in this case does not follow from Klingenberg's Theorem and which is needed to show the uniform bound of the curvature. This concluded the proof of Theorem 3.1.1.

In an alternative approach [Ham95c], Hamilton showed when $\mathcal{M} = \mathbb{S}^2$, the monotonicity of the isoperimetric constant of the sphere, using the curve shrinking flow on a Ricci flow background. This can be utilised to show a uniform lower bound of the injectivity radius which allows to apply singularity analysis [Ham95b] to conclude convergence of the rescaled flow to the shrinking sphere. Alternatively, one could use techniques from Perelman (e.g. the monoticity of the $\mathcal{W}$-entropy under Ricci flow [Per02, §3]) in order to exclude a local collapse of the metric.

Note, that there is a proof of Theorem 3.1.1 which does not rely on the uniformisation theorem;* on the contrary, the theorem recovers the uniformisation theorem for compact surfaces. See [CK04, §5] or [Gie07] for a detailed survey on the results and techniques mentioned above.

Overview

Since the original proof of Hamilton and Chow is quite intricate, and since we are only interested in the case of the simply connected sphere $\mathcal{M} = \mathbb{S}^2$, we are going to present here a much shorter and very elegant variant [BSY94] by J. Bartz, Michael Struwe and Rugang Ye. The key to this approach is a uniform gradient estimate for the Ricci flow's conformal factor (→ Theorem 3.2.3), whose proof is an adaption of those in [Ye94] for the Yamabe flow in higher dimensions. It relies on the *Aleksandrov reflection method*, which was first exploited to this extent in an elliptic setting by Basilis Gidas, Wei-Ming Ni and Louis Nirenberg [GNN79, §4]. However, it can be extended to conformally invariant, uniformly parabolic flows since it just requires the strong maximum principle and the Hopf boundary point lemma. Chow proved a more general variant [Cho97, §3], which relies only on the weak maximum principle and hence extends even to degenerate parabolic, conformally invariant flows. This will be the contents of Subsection 3.2.2 and the proof of Theorem 3.2.3.

The rest of the chapter follows [BSY94]: The gradient estimate implies a Harnack inequality (→ Corollary 3.2.4) which can be exploited to show uniform C^k-bounds of the Ricci flow's conformal factor (→ Proposition 3.2.5) and curvature (→ Corollary 3.2.6). These ensure long-time existence and uniqueness. The convergence of the curvature (→ §3.4.1) is also a consequence of the *a priori* C^k-bounds for the flows's conformal factor. Finally, an argument as in Hamilton's original approach in [Ham88, §9] yields

*In a short note [CLT06], Xiuxiong Chen, Peng Lu and Gang Tian showed without using the uniformisation theorem, that the only shrinking gradient Ricci soliton in two dimensions is the spherical space form.

exponential and smooth convergence of the normalised Ricci flow to the round sphere ($\rightarrow$ §3.4.2 and §3.4.3).

Note that more recently, STRUWE gave an even more elementary and shorter self-containing proof of existence and convergence to the round sphere [Str02]. Instead of using the maximum principle, it is based on elementary integral estimates. This approach was inspired by XIUXIONG CHEN's work on the analogical problem for the Calabi flow [Che01].

3.2 *A priori* estimates

3.2.1 Normalised Ricci flow on the 2-sphere

Given a Ricci flow $\big(g(t)\big)_{t\in[0,\tau]}$ we choose $\psi(t) = \frac{1}{2(T-t)}$ and $t_0 = 0$ in Proposition 2.2.3 where $T = \frac{1}{8\pi}\operatorname{vol}_{g(0)}\mathbb{S}^2$, and obtain $\bar\kappa = 1$ in (2.11), i.e.

$$\frac{\partial}{\partial \bar t}\bar g(\bar t) = -2(\mathrm{K}_{\bar g(\bar t)} - 1)\bar g(\bar t). \tag{3.3}$$

Consequently, we have the conversions between normalised and unnormalised Ricci flow

$$\begin{aligned} \bar g\big(\bar t(t)\big) &= \frac{1}{2(T-t)}g(t) &&\text{and}\quad \bar t(t) = \frac{1}{2}\log\frac{T}{T-t} &&\text{for all } t\in[0,\tau] \\ g\big(t(\bar t)\big) &= 2T\mathrm{e}^{-2\bar t}\bar g(\bar t) &&\text{and}\quad t(\bar t) = T\left(1-\mathrm{e}^{-2\bar t}\right) &&\text{for all } \bar t\in[0,\bar\tau] \end{aligned} \tag{3.4}$$

with $\bar\tau = \frac{1}{2}\log\frac{T}{T-\tau}$. Note, that in contrast to the classical theory for Ricci flows on compact surfaces ($\rightarrow$ Theorem 3.1.1), where $\bar\kappa = ⨍_{\mathcal{M}}\mathrm{K}_{g(0)}\,\mathrm{d}\mu_{g(0)}$ and the initial metrics of normalised and unnormalised flow do not differ, here the normalised metric is initially rescaled by $\frac{1}{2T}$. Consequently with the above choice of T, we have fixed the initial volume to be that of the round unit sphere, i.e. $\operatorname{vol}_{\bar g(0)}\mathbb{S}^2 = 4\pi$. By integrating the evolution of the volume under normalised Ricci flow ($\rightarrow$ Proposition 2.2.6) and using the GAUSS-BONNET Theorem A.1.1, one observes that with the above choice of $T = \frac{1}{8\pi}\operatorname{vol}_{g(0)}\mathbb{S}^2$ the volume stays constant under the normalised flow

$$\begin{aligned} &\operatorname{vol}_{\bar g(\bar t)}\mathbb{S}^2 \overset{(2.15)}{=} \operatorname{vol}_{\bar g(0)}\mathbb{S}^2 - 2\bar t\int_{\mathbb{S}^2}\big(\mathrm{K}_{\bar g(\bar t)} - 1\big)\,\mathrm{d}\mu_{\bar g(\bar t)} \overset{(A.1)}{=} 4\pi(1-2\bar t) + 2\bar t\operatorname{vol}_{\bar g(\bar t)}\mathbb{S}^2, \\ \Longrightarrow\quad &\operatorname{vol}_{\bar g(\bar t)}\mathbb{S}^2 = 4\pi \qquad\qquad \text{for all } \bar t\in\big[0,\bar\tau\big]. \end{aligned} \tag{3.5}$$

If we write $\mathrm{e}^{2\bar u(\bar t)}g_{\mathbb{S}} = \bar g(\bar t)$ for all $\bar t\in[0,\bar\tau]$, then

$$\frac{\partial}{\partial\bar t}\bar u = \mathrm{e}^{-2\bar u}\left(\Delta_{g_{\mathbb{S}}}\bar u - 1\right) + 1. \tag{3.6}$$

3.2.2 Aleksandrov reflection method on the n-sphere

In order to discuss the *Aleksandrov reflection* on the n-sphere $\big(\mathbb{S}^n, \mathrm{e}^{2u}g_{\mathbb{S}}\big)$, we fix a point $p\in\mathbb{S}^n$ and consider the embedded unit sphere $\mathbb{S}^n\subset\big(\mathbb{R}^{n+1},\langle\cdot,\cdot\rangle\big)$ such that $\boldsymbol{p} = (0,\dots,0,1)$ is the north pole. We will correlate points $q\in\mathbb{S}^n$ with unit-length vectors in $\boldsymbol{q}\in\big(\mathbb{R}^{n+1},\langle\cdot,\cdot\rangle\big)$. Further, let $\mathbb{S}^n_{p,+} := \big\{q\in\mathbb{S}^n : \langle\boldsymbol{q},\boldsymbol{p}\rangle\geq 0\big\}$ denote the half sphere around p and $\partial\mathbb{S}^n_{p,+} := \big\{q\in\mathbb{S}^n : \langle\boldsymbol{q},\boldsymbol{p}\rangle = 0\big\}$ the associated equator. This way, we can assign a unique vector field $\boldsymbol{E}_p\in\mathfrak{X}\big(\mathbb{S}^n\setminus\{p,-p\}\big)$ such that it coincides with $\boldsymbol{p}$ on the equator and extends via parallel transport with respect to the round metric $g_{\mathbb{S}}$

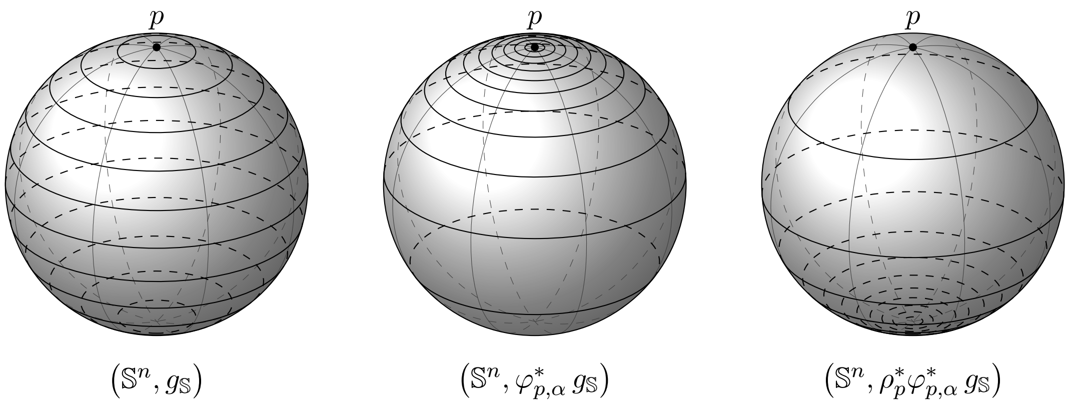


Figure 3.1: *Aleksandrov reflection* on the round sphere $(\mathbb{S}^n, g_{\mathbb{S}})$ with $\alpha = \frac{1}{4}$.

to the whole sphere punctured at its poles $\mathbb{S}^n \setminus \{p, -p\}$. The associated *stereographic projection map* $\sigma_p : \mathbb{S}^n \setminus \{p\} \to \mathbb{R}^n \times \{\mathbf{0}\} \subset \mathbb{R}^{n+1}$ and its inverse $\sigma_p^{-1} : \mathbb{R}^n \times \{\mathbf{0}\} \to \mathbb{S}^n \setminus \{p\}$ are given by

$$\sigma_p(q) = \frac{\boldsymbol{q} - \langle \boldsymbol{p}, \boldsymbol{q}\rangle \boldsymbol{p}}{1 - \langle \boldsymbol{p}, \boldsymbol{q}\rangle} \qquad \text{and} \qquad \sigma_p^{-1}(\boldsymbol{y}) = \frac{2\boldsymbol{y} + \big(|\boldsymbol{y}|^2 - 1\big)\,\boldsymbol{p}}{|\boldsymbol{y}|^2 + 1}. \tag{3.7}$$

Now we can specify two conformal diffeomorphisms on $(\mathbb{S}^n, g)$: For $\alpha \in (0, \infty)$ we define the *conformal dilation* $\varphi_{p,\alpha} : \mathbb{S}^n \to \mathbb{S}^n$ and the *reflection about the equator* $\rho_p : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\begin{aligned} \varphi_{p,\alpha}(q) &:= \sigma_p^{-1}\big(\alpha \cdot \sigma_p(q)\big) = \frac{2\alpha \boldsymbol{q} + \big(\alpha^2 - 1 + (\alpha - 1)^2 \langle \boldsymbol{p}, \boldsymbol{q}\rangle\big)\,\boldsymbol{p}}{\alpha^2 + 1 + (\alpha^2 - 1)\,\langle \boldsymbol{p}, \boldsymbol{q}\rangle} \\ \text{and} \qquad \rho_p(q) &:= \boldsymbol{q} - 2\langle \boldsymbol{p}, \boldsymbol{q}\rangle \boldsymbol{p} \end{aligned} \tag{3.8}$$

for all $q \in \mathbb{S}^n$. For $\alpha < 1$ the conformal dilation $\varphi_{p,\alpha}$ takes points closer to the antipode of p, while the pullback metric $\varphi_{p,\alpha}^* g$ is concentrated near p. This is illustrated in Figure 3.1, where the latitudes on each sphere are equidistant (with distance $\frac{\pi}{12}$). Finally using $|\sigma_p(q)|^2 = \frac{1 + \langle \boldsymbol{p}, \boldsymbol{q}\rangle}{1 - \langle \boldsymbol{p}, \boldsymbol{q}\rangle}$, we calculate the conformal factor $u_{p,\alpha} \in \mathrm{C}^\infty(\mathbb{S}^n)$ of the pullback metric

$$\begin{aligned} \mathrm{e}^{2u_{p,\alpha}(q)}\, g_{\mathbb{S}} := \varphi_{p,\alpha}^* \left(\mathrm{e}^{2u}\, g_{\mathbb{S}}\right)_q &= \sigma_p^* \left(\alpha^2 \mathrm{e}^{2u \circ \sigma_p^{-1}(\alpha \boldsymbol{y})} \frac{4}{\left(|\alpha \boldsymbol{y}|^2 + 1\right)^2}\, g_{\mathbb{E}} \right)_{\boldsymbol{y} = \sigma_p(q)} \\ &= \mathrm{e}^{2u \circ \varphi_{p,\alpha}(q)} \frac{4\alpha^2}{\left(\alpha^2 |\sigma_p(q)|^2 + 1\right)^2} \cdot \frac{\left(|\sigma_p(q)|^2 + 1\right)^2}{4}\, g_{\mathbb{S}} \\ &= \mathrm{e}^{2u \circ \varphi_{p,\alpha}(q)} \frac{4\alpha^2}{\left(\alpha^2 + 1 + (\alpha^2 - 1)\langle \boldsymbol{p}, \boldsymbol{q}\rangle\right)^2}\, g_{\mathbb{S}} \end{aligned} \tag{3.9}$$

and hence the pushforward of the vector field $\boldsymbol{E}_p$

$$\left(\varphi_{p,\alpha}\right)_{*q} \boldsymbol{E}_p(q) = \frac{2\alpha}{\alpha^2 + 1 + (\alpha^2 - 1)\langle \boldsymbol{p}, \boldsymbol{q}\rangle}\, \boldsymbol{E}_p\big(\varphi_{p,\alpha}(q)\big) \quad \text{for all } q \in \mathbb{S}^n \setminus \{p, -p\}. \tag{3.10}$$

Lemma 3.2.1 (Aleksandrov reflection)**.** *For any sphere $(\mathbb{S}^n, g)$ there exists a constant*

$\alpha_0 \in (0,1)$ *such that for all* $\alpha \in (0, \alpha_0]$ *and points* $p \in \mathbb{S}^n$*,*

$$\varphi_{p,\alpha}^* g \geq \rho_p^* \varphi_{p,\alpha}^* g \quad \text{on } \mathbb{S}^n_{p,+}. \tag{3.11}$$

Although the result seems to be quite obvious, its proof is lengthy and technical, even though it is elementary. Therefore we omit it and refer to [Cho97, Proposition 3.1] for details. Note, if $g = \mathrm{e}^{2u} g_{\mathbb{S}}$ in Lemma 3.2.1, then we can rephrase (3.11) in terms of the conformal factor u using the notation (3.9),

$$u_{p,\alpha} \geq u_{p,\alpha} \circ \rho_p \qquad \text{on } \mathbb{S}^n_{p,+}. \tag{3.11*}$$

The *Aleksandrov reflection* implies a gradient bound of the conformal factor.

Proposition 3.2.2 ([Cho97, pp. 403–405])**.** *For some* $\alpha_0 \in (0,1)$ *let* $\mathrm{e}^{2u} g_{\mathbb{S}}$ *satisfy the Aleksandrov reflection* (3.11) *for all* $\alpha \in (0, \alpha_0]$ *and* $p \in \mathbb{S}^n$*. Then,*

$$\left| \nabla u \right|_{g_{\mathbb{S}}} \leq \frac{1 - \alpha_0^2}{2\alpha_0}. \tag{3.12}$$

Proof. For any $p \in \mathbb{S}^n$, the Aleksandrov reflection (3.11*) and the gradient of $u_{p,\alpha}$ imply for all $q \in \partial \mathbb{S}^n_{p,+}$ on the equator and $\alpha \in (0, \alpha_0]$

$$\begin{aligned}
0 &\leq \big\langle \nabla u_{p,\alpha}(q), \boldsymbol{p} \big\rangle \\
&= \mathrm{d}\big(u \circ \varphi_{p,\alpha}\big)_q \boldsymbol{E}_p(q) + \frac{\left(1 - \alpha^2\right) \big\langle \nabla \langle \boldsymbol{p}, \boldsymbol{q} \rangle, \boldsymbol{p} \big\rangle}{\alpha^2 + 1 + (\alpha^2 - 1) \langle \boldsymbol{p}, \boldsymbol{q} \rangle} \\
&= (\mathrm{d}u)_{\varphi_{p,\alpha}(q)} \left(\varphi_{p,\alpha}\right)_{*q} \boldsymbol{E}_p(q) + \frac{\left(1 - \alpha^2\right)}{\alpha^2 + 1} \\
&= \frac{2\alpha}{\alpha^2 + 1} \big\langle \nabla u\big(\varphi_{p,\alpha}(q)\big), \boldsymbol{E}_p\big(\varphi_{p,\alpha}(q)\big) \big\rangle + \frac{\left(1 - \alpha^2\right)}{\alpha^2 + 1},
\end{aligned}$$

where we used (3.9), $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0$, $\big\langle \nabla \langle \boldsymbol{p}, \boldsymbol{q} \rangle, \boldsymbol{p} \big\rangle = 1$ and (3.10). Therefore, for any $p, q \in \mathbb{S}^n$ with $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0$ we have

$$\Big\langle \nabla u, \boldsymbol{E}_p \Big\rangle \big(\varphi_{p,\alpha_0}(q)\big) \geq -\frac{1 - \alpha_0^2}{2\alpha_0}. \tag{3.13}$$

In order to transform this estimate into (3.12), we fix a point $p_0 \in \mathbb{S}^n$ and consider the two subsets

$$\begin{aligned}
Q_\alpha(p_0) &:= \Big\{ q \in \mathbb{S}^n : \varphi_{p,\alpha}(q) = p_0 \text{ for some } p \in \mathbb{S}^n \text{ with } \langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0 \Big\} \\
&\overset{(3.8)}{=} \left\{ q \in \mathbb{S}^n : \langle \boldsymbol{q}, \boldsymbol{p}_0 \rangle = \frac{2\alpha}{\alpha^2 + 1} \right\} \\
\text{and} \quad P_\alpha(p_0) &:= \Big\{ p \in \mathbb{S}^n : \varphi_{p,\alpha}(q) = p_0 \text{ for some } q \in \mathbb{S}^n \text{ with } \langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0 \Big\} \\
&\overset{(3.8)}{=} \left\{ p \in \mathbb{S}^n : \langle \boldsymbol{p}, \boldsymbol{p}_0 \rangle = \frac{\alpha^2 - 1}{\alpha^2 + 1} \right\},
\end{aligned}$$

which are $(n-1)$-spheres centered at p_0 ($\rightarrow$ Figure 3.2). Therefore, for every unit-length vector $\boldsymbol{v} \in \mathrm{T}\mathbb{S}^n_{p_0}$ there are unique points $p_{\boldsymbol{v}} \in P_{\alpha_0}(p_0)$ and $q_{\boldsymbol{v}} \in Q_{\alpha_0}(p_0)$ such that $\boldsymbol{E}_{p_{\boldsymbol{v}}}(p_0) = \boldsymbol{v}$ and $\langle \boldsymbol{q}_{\boldsymbol{v}}, \boldsymbol{p}_{\boldsymbol{v}} \rangle = 0$. Because $p_0 \in \mathbb{S}^n$ was chosen arbitrarily, we conclude

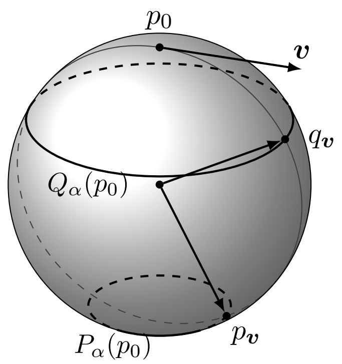


Figure 3.2: The $(n-1)$-spheres $Q_\alpha(p_0)$ and $P_\alpha(p_0)$ for $\alpha = \frac{1}{4}$.

(3.12) since for all unit-length vectors $\boldsymbol{v} \in \mathrm{T}\mathbb{S}^n_{p_0}$ we have

$$\big\langle \nabla u(p_0), \boldsymbol{v} \big\rangle = \Big\langle \nabla u(p_0), \boldsymbol{E}_{p_{\boldsymbol{v}}}(p_0) \Big\rangle$$
$$= \Big\langle \nabla u\big(\varphi_{p_{\boldsymbol{v}},\alpha_0}(q_{\boldsymbol{v}})\big), \boldsymbol{E}_{p_{\boldsymbol{v}}}\big(\varphi_{p_{\boldsymbol{v}},\alpha_0}(q_{\boldsymbol{v}})\big) \Big\rangle \overset{(3.13)}{\geq} -\frac{1-\alpha_0^2}{2\alpha_0}. \qquad \square$$

3.2.3 Gradient estimate

Returning to the normalised Ricci flow on the two-sphere, we can exploit the Aleksandrov reflection method in order to show a *uniform* bound to the gradient of the metric's conformal factor which depends only on the initial metric.

Theorem 3.2.3 ([BSY94, Proposition 2.2])**.** *For some $\bar{T} > 0$ let $\big(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\big)_{\bar{t}\in[0,\bar{T}]}$ be a normalised Ricci flow on $\mathbb{S}^2$. Then there exists a constant $C = C\big(u(0)\big) < \infty$ such that*

$$\Big|\nabla \bar{u}(\bar{t})\Big|_{g_{\mathbb{S}}} \leq C \qquad \textit{for all } \bar{t} \in [0, \bar{T}]. \tag{3.14}$$

In contrast to the original variant in [BSY94] we are going to present a proof by Chow which relies only on the weak maximum principle; therefore, it can be generalised to arbitrary weakly parabolic and conformal flows (→ [Cho97, Theorem 3.2]).

Proof. Lemma 3.2.1 applied to $\mathrm{e}^{2\bar{u}(0)} g_{\mathbb{S}}$ yields a constant $\alpha_0 \in (0,1)$ such that for all $\alpha \in (0, \alpha_0]$ and $p \in \mathbb{S}^2$

$$\varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(0)} g_{\mathbb{S}}\right) \geq \rho^*_p \varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(0)} g_{\mathbb{S}}\right) \quad \text{on } \mathbb{S}^2_{p,+}.$$

Note, that both $\varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\right) = \mathrm{e}^{2\bar{u}_{p,\alpha}(\bar{t})} g_{\mathbb{S}}$ and $\rho^*_p \varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\right) = \mathrm{e}^{2(\bar{u}_{p,\alpha}(\bar{t})\circ\rho_p)} g_{\mathbb{S}}$ are again solutions of the normalised Ricci flow (3.3), because $\varphi_{p,\alpha}$ and ρ_p are conformal maps. Since $\rho_p(q) = q$ for all $q \in \partial\mathbb{S}^2_{p,+}$, we can apply a direct comparison principle (Theorem 2.3.1) for solutions of (3.6)† in order to compare $\bar{u}_{p,\alpha}(\bar{t})$ and $\bar{u}_{p,\alpha}(\bar{t}) \circ \rho_p$ on $\mathbb{S}^2_{p,+}$, and conclude that for all $\alpha \in (0, \alpha_0]$ and $(\bar{t}, p) \in [0, \bar{T}] \times \mathbb{S}^2$

$$\varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\right) \geq \rho^*_p \varphi^*_{p,\alpha}\left(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\right) \quad \text{on } \mathbb{S}^2_{p,+}.$$

† By choosing a chart on the $\mathbb{S}^2_{p,+}$, it is not hard to see that Theorem 2.3.1 also applies to solutions of the normalised equation with a spherical background metric (3.3).

By virtue of Proposition 3.2.2, this is equivalent to (3.14) with $C := \frac{1-\alpha_0^2}{2\alpha_0}$. $\square$

Integrating (3.14) along a distance realising geodesic from the minimum to the maximum of $\bar{u}(\bar{t})$ yields the following Harnack inequality.

Corollary 3.2.4 ([BSY94, Eq. (16)]). *For some $\bar{T} > 0$ let $\big(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\big)_{\bar{t}\in[0,\bar{T}]}$ be a normalised Ricci flow on $\mathbb{S}^2$. Then there exists a constant $C = C\big(u(0)\big) < \infty$ such that*

$$\max_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \leq C \min_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \qquad \textit{for all } \bar{t} \in [0, \bar{T}]. \tag{3.15}$$

Proof. Fix $\bar{t} \in [0, \bar{T}]$. Let $p_+, p_- \in \mathbb{S}^2$ be the maximum and minimum of $\bar{u}(\bar{t})$, i.e., $\bar{u}\big(\bar{t}, p_\pm\big) = \pm \max_{p\in\mathbb{S}^2} \pm\bar{u}(\bar{t}, p)$, and let $\gamma \in \mathrm{C}^\infty\big([0,1], \mathbb{S}^2\big)$ be the $g_{\mathbb{S}}$-geodesic realising the distance from $p_- = \gamma(0)$ to $p_+ = \gamma(1)$. With the constant $\tilde{C} < \infty$ from Theorem 3.2.3 we estimate

$$\bar{u}(\bar{t}, p_+) - \bar{u}(\bar{t}, p_-) = \int_0^1 \Big\langle \nabla\bar{u}\big(\bar{t}, \gamma(s)\big), \dot{\gamma}(s) \Big\rangle \, \mathrm{d}s \leq \pi\tilde{C},$$

and (3.15) follows with $C = \mathrm{e}^{2\pi\tilde{C}}$. $\square$

3.2.4 C^k bounds

This Harnack inequality allows us to conclude a uniform C^0-bound for the normalised Ricci flow's conformal factor using the constancy of the volume, which can be bootstrapped into C^k estimates.

Proposition 3.2.5 ([BSY94, p. 480]). *For some $\bar{T} > 0$ let $\big(\mathrm{e}^{2\bar{u}(\bar{t})} g_{\mathbb{S}}\big)_{\bar{t}\in[0,\bar{T}]}$ be a normalised Ricci flow on $\mathbb{S}^2$. Then, for each $k \in \mathbb{N}_0$ there exists a constant $C_k = C_k\big(k, \bar{u}(0)\big) < \infty$ such that*

$$\Big\|\bar{u}(\bar{t})\Big\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \leq C_k \qquad \textit{for all } \bar{t} \in [0, \bar{T}]. \tag{3.16}$$

Proof. Using the Harnack inequality (Corollary 3.2.4) with constant $C \in [1, \infty)$, we estimate

$$\begin{aligned} 1 \overset{(3.5)}{=} \frac{\int_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \, \mathrm{d}\mu_{g_{\mathbb{S}}}}{4\pi} \leq \max_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \overset{(3.15)}{\leq} C \min_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \leq C \, \frac{\int_{\mathbb{S}^2} \mathrm{e}^{2\bar{u}(\bar{t})} \, \mathrm{d}\mu_{g_{\mathbb{S}}}}{4\pi} \overset{(3.5)}{=} C, \\ \implies \qquad \Big|\bar{u}(\bar{t})\Big| \leq C_0 := \frac{1}{2}\log C \qquad \text{for all } \bar{t} \in \big[0, \bar{T}\big]. \end{aligned} \tag{3.17}$$

Along with (3.14) we get a Hölder bound on $\bar{u}(\bar{t})$, which can be bootstrapped using parabolic Schauder estimates (Corollary B.2.2) to obtain the C^k-estimates (3.16). $\square$

Using the relation $\mathrm{K}_g = -\mathrm{e}^{-2u}\left(\Delta_{g_{\mathbb{S}}} u - 1\right)$ for $g = \mathrm{e}^{2u} g_{\mathbb{S}}$ ($\to$ Lemma 2.2.1), we obtain also uniform C^k-estimates for the curvature.

Corollary 3.2.6. *For some $\bar{T} > 0$ let $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\bar{T}]}$ be a normalised Ricci flow on the sphere $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$. Then, for each $k \in \mathbb{N}_0$ there exists a constant $C_k = C_k\big(k, \bar{g}(0)\big) < \infty$ such that*

$$\Big\|\mathrm{K}_{\bar{g}(\bar{t})}\Big\|_{\mathrm{C}^k\left(\mathbb{S}^2, \bar{g}(\bar{t})\right)} \leq C_k \qquad \textit{for all } \bar{t} \in [0, \bar{T}]. \tag{3.18}$$

Using (3.4) we convert (3.18) into an *a priori* curvature bound of the unnormalised flow.

Remark 3.2.7. For some $\tau > 0$ let $\big(g(t)\big)_{t\in[0,\tau]}$ be a Ricci flow on the sphere $(\mathbb{S}^2, g_{\mathbb{S}})$. Then there exists a constant $C = C\big(g(0)\big) < \infty$ such that

$$\left|\mathrm{K}_{g(t)}\right| \leq \frac{C}{T-t} \qquad \text{for all } t \in [0,\tau] \tag{3.19}$$

where $T = \frac{1}{8\pi}\operatorname{vol}_{g(0)}\mathbb{S}^2$.

3.3 Existence and uniqueness

Lemma 3.3.1. *Let $\big(g(t)\big)_{t\in[0,\hat{T})}$ be a Ricci flow on the sphere $\mathbb{S}^2$. Then $\hat{T} \leq T = \frac{1}{8\pi}\operatorname{vol}_{g(0)}\mathbb{S}^2$ and*

$$\operatorname{vol}_{g(t)}\mathbb{S}^2 = 8\pi(T-t) \qquad \textit{for all } t \in [0,\hat{T}). \tag{3.20}$$

Proof. Integrating (2.15) and using the Gauss-Bonnet Theorem A.1.1, we obtain

$$\operatorname{vol}_{g(t)}\mathbb{S}^2 = \operatorname{vol}_{g(0)}\mathbb{S}^2 - 2\int_0^t\int_{\mathbb{S}^2}\mathrm{K}_{g(\tau)}\,\mathrm{d}\mu_{g(\tau)}\,\mathrm{d}\tau = 8\pi T - 4\pi\,\chi(\mathbb{S}^2)\,t \quad \text{for all } t\in[0,\hat{T}).$$

Hence, any solution extinguishes no later than $T = \frac{1}{8\pi}\operatorname{vol}_{g_0}\mathbb{S}^2$. □

Theorem 3.3.2 (Long-time existence and uniqueness of Ricci flows on the 2-sphere)**.** *Let g_0 be a smooth Riemannian metric on the sphere $\mathbb{S}^2$. Then there exists a unique Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ with $g(0) = g_0$ up to a maximal time $T = \frac{1}{8\pi}\operatorname{vol}_{g_0}\mathbb{S}^2$. Moreover, we have the volume equation (3.20) for all $t \in [0,T)$.*

Proof. Since $\mathbb{S}^2$ is compact, Hamilton's Theorem 2.4.1 provides a unique Ricci flow $\big(g(t)\big)_{t\in[0,T_1]}$ with $g(0) = g_0$ up to a time $T_1 \in (0,T)$. Using Corollary 2.4.2 to gradually continue the flow, we obtain an extended solution $\big(g(t)\big)_{t\in[0,T_\infty)}$ up to a maximal time $T_\infty \leq T$. Assuming $T_\infty < T$, by (3.19) there is a constant $C < \infty$ such that Gaussian curvature $\left|\mathrm{K}_{g(t)}\right| \leq \frac{C}{T-T_\infty} =: \kappa < \infty$ is bounded uniformly for all $t \in [0,T_\infty)$. Again by Corollary 2.4.2 there is a constant $\tau > 0$ depending only on κ such that we can uniquely extend the solution $\big(g(t)\big)_{t\in[0,T_\infty-\tau/2]}$ to exist for all $t \in [0, T_\infty + \tau/2]$, contradicting the maximality of T_∞. □

Note, that alternatively one could apply directly standard parabolic existence theory to the evolution equation (3.6) for the conformal factor u, because due to the compact domain, the equation is not degenerated and any solution stays bounded by virtue of the *a priori* estimates (Proposition 3.2.5). An entirely trivial consequence of the above uniqueness result is:

Corollary 3.3.3. *Any Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ on the sphere $\mathbb{S}^2$ is both complete and maximally stretched.*

3.4 Long-time behaviour

3.4.1 Convergence of the curvature

Lemma 3.4.1 ([BSY94, Eq. 19]). *Let $\big(\bar g(\bar t)\big)_{\bar t\in[0,\infty)}$ be a normalised Ricci flow on the sphere $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$. Then there is a constant $C = C\big(\bar g(0)\big) < \infty$ such that*

$$\int_0^\infty \left\| \mathrm{K}_{\bar g(\bar t)} - 1 \right\|^2_{\mathrm{L}^2\left(\mathbb{S}^2,\, \bar g(\bar t)\right)} \,\mathrm{d}\bar t \le C. \tag{3.21}$$

Proof. Writing $\bar g(\bar t) = \mathrm{e}^{2\bar u(\bar t)} g_{\mathbb{S}}$ for all $\bar t \in [0,\infty)$, from (3.6) we calculate

$$\begin{aligned}
\int_{\mathbb{S}^2} \left(\mathrm{K}_{\bar g} - 1\right)^2 \mathrm{d}\mu_{\bar g} &= \int_{\mathbb{S}^2} \frac{\partial \bar u}{\partial \bar t}\frac{\partial \bar u}{\partial \bar t}\, \mathrm{e}^{2\bar u}\, \mathrm{d}\mu_{g_{\mathbb{S}}} \\
&= \int_{\mathbb{S}^2} \left(\Delta_{g_{\mathbb{S}}} \bar u - 1 + \mathrm{e}^{2\bar u}\right) \frac{\partial \bar u}{\partial \bar t}\, \mathrm{d}\mu_{g_{\mathbb{S}}} \\
&= -\int_{\mathbb{S}^2} \left(\left\langle \nabla \bar u, \frac{\partial}{\partial \bar t} \nabla \bar u \right\rangle_{g_{\mathbb{S}}} + \frac{\partial \bar u}{\partial \bar t} \right) \mathrm{d}\mu_{g_{\mathbb{S}}} + \frac{\mathrm{d}}{\mathrm{d}\bar t} \int_{\mathbb{S}^2} \frac{1}{2}\, \mathrm{d}\mu_{\bar g} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\bar t} \int_{\mathbb{S}^2} \left(\frac{1}{2} \left|\nabla \bar u\right|^2_{g_{\mathbb{S}}} + \bar u \right) \mathrm{d}\mu_{g_{\mathbb{S}}} =: -\frac{\mathrm{d}}{\mathrm{d}\bar t} E(\bar t),
\end{aligned} \tag{3.22}$$

using the constancy of the volume of the normalised flow. To show (3.21) we integrate (3.22) and estimate using Proposition 3.2.5 with constants $C_0, C_1 \in (0,\infty)$ depending only on $\bar g(0)$

$$\int_0^\infty \left\| \mathrm{K}_{\bar g(\bar t)} - 1 \right\|^2_{\mathrm{L}^2\left(\mathbb{S}^2,\, \bar g(\bar t)\right)} \mathrm{d}\bar t \le E(0) - \sup_{\bar t \in [0,\infty)} E(\bar t) \le C = C\big(C_0, C_1\big) < \infty. \qquad \square$$

Corollary 3.4.2 ([BSY94, p. 481]). *Let $\big(\bar g(\bar t)\big)_{\bar t\in[0,\infty)}$ be a normalised Ricci flow on $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$. Then*

$$\sup_{\mathbb{S}^2} \left| \mathrm{K}_{\bar g(\bar t)} - 1 \right| \quad \xrightarrow{\bar t \to \infty} \quad 0. \tag{3.23}$$

Proof. Combining Corollary 3.2.6 with the evolution (2.14) of $\mathrm{K}_{\bar g(\bar t)}$ yields a uniform (in $\bar t$) bound on $\frac{\partial}{\partial \bar t} \mathrm{K}_{\bar g(\bar t)}$. Hence, there is a constant $C = C\big(\bar g(0)\big) \in (0,\infty)$ such that

$$\left| \frac{\partial}{\partial \bar t} \left\| \mathrm{K}_{\bar g(\bar t)} - 1 \right\|^2_{\mathrm{L}^2\left(\mathbb{S}^2, \bar g(\bar t)\right)} \right| \le C. \tag{3.24}$$

This gradient bound and Lemma 3.4.1 imply

$$\lim_{\bar t \to \infty} \left\| \mathrm{K}_{\bar g(\bar t)} - 1 \right\|^2_{\mathrm{L}^2\left(\mathbb{S}^2, \bar g(\bar t)\right)} = 0. \tag{3.25}$$

Using again the uniform C^k estimates from Corollary 3.2.6 we can interpolate (using e.g. [GT98, Theorem 7.28]) to obtain the convergence with respect to the $\mathrm{W}^{1,3}$-norm which embeds into C^0 (e.g. [GT98, Theorem 7.10]) concluding (3.23). $\square$

3.4.2 Potential of the curvature

Lemma 3.4.3 ([Ham88, §9])**.** *Let $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\infty)}$ be a normalised Ricci flow on $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$ and let the potential $\phi \in \mathrm{C}^\infty\big([0,\infty)\times\mathbb{S}^2\big)$ be a solution of*

$$\Delta_{\bar{g}(\bar{t})}\phi(\bar{t}) = 1 - \mathrm{K}_{\bar{g}(\bar{t})} \qquad \text{for all } \bar{t}\in[0,\infty). \tag{3.26}$$

The norm of the trace-free part of its Hessian $M_{\bar{g}} = \mathrm{Hess}_{\bar{g}}\,\phi - \frac{1}{2}\Delta_{\bar{g}}\phi\cdot\bar{g}$ evolves under normalised Ricci flow as

$$\frac{\partial}{\partial\bar{t}}\big|M_{\bar{g}}\big|_{\bar{g}}^2 = \Delta_{\bar{g}}\big|M_{\bar{g}}\big|_{\bar{g}}^2 - 2\,\big|{}^{\bar{g}}\nabla M_{\bar{g}}\big|_{\bar{g}}^2 - 4\mathrm{K}_{\bar{g}}\,\big|M_{\bar{g}}\big|_{\bar{g}}^2. \tag{3.27}$$

Proof. For the sake of readability, we write all geometric quantities like ∇, Hess, Δ, $|\cdot|$, $\langle\cdot,\cdot\rangle$, ϕ's trace-free Hessian M and the Gaussian curvature K with respect to the time-dependent metric $\bar{g}(\bar{t})$. We start deriving the evolution of the potential ϕ

$$\begin{aligned}\Delta\frac{\partial\phi}{\partial\bar{t}} &= \frac{\partial}{\partial\bar{t}}\Delta\phi - 2(\mathrm{K}-1)\Delta\phi = -\big(\Delta\mathrm{K} + 2\mathrm{K}(\mathrm{K}-1)\big) + 2(1-\mathrm{K})^2\\ &= \Delta(1-\mathrm{K}) + 2(1-\mathrm{K}) = \Delta\big(\Delta\phi + 2\phi\big).\end{aligned} \tag{3.28}$$

Since the only harmonic functions on the compact sphere are the constant ones, ϕ and its evolution (3.28) are defined up to some time dependent constant, which we do not have to refine, because we are only interested in its trace-free Hessian M. For some vector fields $\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}(\mathbb{S}^2)$ we calculate using the evolution of the Levi-Cevita connection (2.13) and the Ricci identity to commute covariant derivatives

$$\begin{aligned}\frac{\partial}{\partial\bar{t}}M(\boldsymbol{X},\boldsymbol{Y}) &= \frac{\partial}{\partial\bar{t}}\left(\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\phi - \frac{1}{2}(1-\mathrm{K})\langle\boldsymbol{X},\boldsymbol{Y}\rangle\right)\\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\frac{\partial\phi}{\partial\bar{t}} - \mathrm{d}\phi\left(\frac{\partial}{\partial\bar{t}}\nabla_{\boldsymbol{X}}\boldsymbol{Y}\right) + \frac{1}{2}\frac{\partial\mathrm{K}}{\partial\bar{t}}\langle\boldsymbol{X},\boldsymbol{Y}\rangle - \frac{1}{2}(1-\mathrm{K})\frac{\partial}{\partial\bar{t}}\langle\boldsymbol{X},\boldsymbol{Y}\rangle\\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}(\Delta\phi+2\phi) + \big((\boldsymbol{Y}\mathrm{K})(\boldsymbol{X}\phi) + (\boldsymbol{X}\mathrm{K})(\boldsymbol{Y}\phi) - \langle\nabla\mathrm{K},\nabla\phi\rangle\langle\boldsymbol{X},\boldsymbol{Y}\rangle\big)\\ &\qquad + \frac{1}{2}\big(\Delta\mathrm{K} + 2\mathrm{K}(\mathrm{K}-1) - 2(\mathrm{K}-1)^2\big)\langle\boldsymbol{X},\boldsymbol{Y}\rangle\\ &= \Bigg(\mathrm{Hess}\,\Delta\phi + \nabla\phi\otimes\nabla\mathrm{K} + \nabla\mathrm{K}\otimes\nabla\phi\\ &\qquad - \langle\nabla\mathrm{K},\nabla\phi\rangle g + \frac{1}{2}\Delta\mathrm{K}\,g + 2M\Bigg)(\boldsymbol{X},\boldsymbol{Y})\\ &= \left(\Delta\,\mathrm{Hess}\,\phi - \frac{1}{2}\Delta(1-\mathrm{K})g - 4\mathrm{K}M + 2M\right)(\boldsymbol{X},\boldsymbol{Y})\\ &= \Big(\Delta M + 2(1-2\mathrm{K})M\Big)(\boldsymbol{X},\boldsymbol{Y}).\end{aligned}$$

Consequently, we conclude

$$\begin{aligned}\frac{\partial}{\partial\bar{t}}\big|M\big|^2 &= 2\left\langle M,\frac{\partial}{\partial\bar{t}}M\right\rangle - 4(1-\mathrm{K})\big|M\big|^2 = 2\big\langle M,\Delta M\big\rangle - 4\mathrm{K}\big|M\big|^2\\ &= \big|M\big|^2 - 2\big|\nabla M\big|^2 - 4\mathrm{K}\big|M\big|^2.\end{aligned}$$

□

Proposition 3.4.4. *Let $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\infty)}$ be a normalised Ricci flow on $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$. Then, for all $\eta\in(0,1)$ and for all $k\in\mathbb{N}_0$ there exist constants $C_k = C_k\big(k,\eta,\bar{g}(0)\big)\in(0,\infty)$ such* *that*

$$\left\| M_{\bar g(\bar t)} \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \leq C_k \mathrm{e}^{-2(1-\eta)\bar t} \qquad \textit{for all } \bar t \in [0,\infty). \tag{3.29}$$

Proof. By Corollary 3.4.2, there is a time $\bar t_\eta \in [0,\infty)$ such that $\mathrm{K}_{\bar g(\bar t)} \geq 1 - \eta/2$ for all $\bar t \in \left[\bar t_\eta, \infty\right)$. Then we can estimate the evolution of $M_{\bar g}$ in Lemma 3.4.3 by

$$\frac{\partial}{\partial \bar t} \left| M_{\bar g} \right|^2_{\bar g} \leq \Delta_{\bar g} \left| M_{\bar g} \right|^2_{\bar g} - 4(1 - \eta/2) \left| M_{\bar g} \right|^2_{\bar g} \tag{3.30}$$

for all $\bar t \geq \bar t_\eta$. Since $M_{\bar g}$ is fully determinend by the conformal factor of the metric $\mathrm{e}^{2\bar u} g_{\mathbb{S}} = \bar g$, by Proposition 3.2.5 we have uniform C^k bounds $C_k = C_k\big(k, g(0)\big) < \infty$ such that $\left\| M_{\bar g} \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \leq C_k$. Comparing (3.30) with the associated ordinary differential equation (Theorem 2.3.4) allows us to conclude (3.29) for $k = 0$, i.e.

$$\left| M_{\bar g(\bar t)} \right|_{g_{\mathbb{S}}} \leq C_0 \, \mathrm{e}^{-2(1-\eta/2)\bar t} \qquad \text{for all } \bar t \in [0,\infty), \tag{3.31}$$

using also the uniform equivalence of the metrics $\bar g(\bar t)$ and $g_{\mathbb{S}}$ by the same Proposition 3.2.5. Interpolating those uniform C^k-bounds with the C^0-estimate (3.31) ($\to$ Lemma B.3.1), we obtain the decay estimate for higher derivatives, i.e. (3.29). □

3.4.3 Convergence

Finally, we conclude this chapter showing that the normalised flow converges smoothly to the round sphere. Though the argument is the same as in [Ham88], i.e. we show that the normalised Ricci flow converges to a gradient Ricci soliton ($M_{\bar g} \equiv 0$), due to Corollary 3.4.2 we already know that this limit has constant curvature, whereas Hamilton had to argue why two-dimensional shrinking gradient Ricci solitons are indeed space forms [Ham88, Theorem 10.1].

Theorem 3.4.5. *Let $\big(\bar g(\bar t)\big)_{\bar t \in [0,\infty)}$ be a normalised Ricci flow on $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$. Then for every $\eta \in (0,1)$ and $k \in \mathbb{N}_0$ there is a constant $C = C\big(k, \eta, \bar g(0)\big) \in (0,\infty)$ such that*

$$\left\| \bar g(\bar t) - g_{\mathbb{S}} \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \leq C \mathrm{e}^{-2(1-\eta)\bar t} \qquad \textit{for all } \bar t \in [0,\infty). \tag{3.32}$$

Proof. Let $\phi \in \mathrm{C}^\infty\big([0,\infty) \times \mathbb{S}^2\big)$ be a potential of the curvature of $\bar g$, i.e. a solution of (3.26) and $M_{\bar g}$ its trace-free Hessian. Further, let $\big(\psi_{\bar t}\big)_{\bar t \in [0,\infty)} \in \operatorname{Diff}\big(\mathbb{S}^2\big)$ be the family of diffeomorphisms generated by $-\operatorname{grad}_{\bar g(\bar t)} \phi(\bar t)$. The family of pullback metrics $\tilde g(\bar t) = \psi^*_{\bar t}\, \bar g(\bar t)$ for all $\bar t \in [0,\infty)$ satisfies the following evolution equation

$$\frac{\partial}{\partial \bar t} \tilde g = -\psi^*_{\bar t} \Big(2\big(\mathrm{K}_{\bar g} - 1\big) \bar g + \mathcal{L}_{\operatorname{grad}_{\bar g} \phi}\, \bar g \Big) = -2\psi^*_{\bar t} \Big(-\Delta_{\bar g} \phi \, \bar g + \operatorname{Hess}_{\bar g} \phi \Big) = -2\psi^*_{\bar t} M_{\bar g}. \tag{3.33}$$

By diffeomorphism invariance, the decay estimates of $M_{\bar g}$ (3.29) from Proposition 3.4.4 are also valid for $\psi^*_{\bar t} M_{\bar g}$, hence integrating (3.33) we see that $\tilde g(\bar t)$ converges exponentially fast and smoothly to a unique limit metric $\tilde g(\infty)$; in particular, with constants $\tilde C_k$ from Proposition 3.4.4 we have for all $\vartheta > 0$

$$\begin{aligned} \left\| \tilde g(\bar t) - \tilde g(\bar t + \vartheta) \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} &\leq \int_{\bar t}^{\bar t + \vartheta} \left\| \frac{\partial}{\partial \tau} \tilde g(\tau) \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \mathrm{d}\tau = 2 \int_{\bar t}^{\bar t + \vartheta} \left\| \psi^*_\tau M_{\bar g(\tau)} \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \mathrm{d}\tau \\ &\leq 4(1-\eta)\big(1 - \mathrm{e}^{-2(1-\eta)\vartheta}\big) \tilde C_k \mathrm{e}^{-2(1-\eta)\bar t}. \end{aligned} \tag{3.34}$$

Since $\tilde{g}(\bar{t})$ and $\bar{g}(\bar{t})$ have the same Gaussian curvature for all $\bar{t} \in [0,\infty)$, Corollary 3.4.2 implies that the limit of the modified flow also has constant Gaussian curvature $\mathrm{K}_{\tilde{g}(\infty)} \equiv 1$, i.e. $\tilde{g}(\infty) = g_{\mathbb{S}}$. By diffeomorphism invariance we have the same uniform, smooth and exponential convergence (3.34) also for the original flow $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\infty)}$, and (3.32) follows with $C = 4(1-\eta)\tilde{C}_k$ letting $\vartheta \to \infty$ in (3.34). □

Using the conversion (3.4), we obtain the corresponding result for the unnormalised Ricci flow.

Corollary 3.4.6. *Let $\big(g(t)\big)_{t\in[0,T)}$ be a Ricci flow on $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$ with maximal time $T = \frac{1}{8\pi}\operatorname{vol}_{g(0)}\mathbb{S}^2$. Then for all $\eta \in (0,1)$ and $k \in \mathbb{N}_0$ there are constants $C = C\big(k,\eta,g(0)\big) < \infty$ such that*

$$\left\| \frac{1}{2(T-t)} g(t) - g_{\mathbb{S}} \right\|_{\mathrm{C}^k(\mathbb{S}^2, g_{\mathbb{S}})} \leq C\big(T-t\big)^{1-\eta} \qquad \text{for all } t \in [0,T). \tag{3.35}$$

Chapter 4

Ricci flow on non-compact, simply connected surfaces

After discussing the Ricci flow on compact surfaces and in particular on the 2-sphere, this chapter is about the corresponding results on non-compact but still simply connected surfaces. The synthesis and generalisation to arbitrary surfaces will follow in the next chapter.

By virtue of the uniformisation theorem, a non-compact, simply connected surface $\big(\mathcal{M}^2, g_0\big)$ is conformally equivalent either to the flat plane or the hyperbolic disc. Therefore, as remarked in Section 2.2.1, we may assume without loss of generality that $\mathcal{M}$ is either $\mathbb{C}$ or $\mathbb{D}$ and write $g_0 = \mathrm{e}^{2u_0}|\mathrm{d}z|^2$ for some $u_0 \in \mathrm{C}^\infty(\mathcal{M})$. Moreover, by the conformal invariance of a two-dimensional Ricci flow $\big(g(t)\big)_{t\in[0,T]}$, we can alternatively restrict our attention to its globally defined conformal factor $u \in \mathrm{C}^\infty\big([0,T]\times\mathcal{M}\big)$ with $g(t) = \mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ and its evolution

$$\frac{\partial}{\partial t}u = \mathrm{e}^{-2u}\Delta u. \tag{4.1}$$

The contents of this chapter in the conformally hyperbolic case follow mainly [GT11] by Peter Topping and the author, in which we extend the study of instantaneously complete Ricci flows, which Topping began 2006 [Top10]. In the conformally flat case on the whole plane, we are going to exploit some existing work from the theory of the *logarithmic fast diffusion equation* which corresponds to (4.1) as we are going to clarify in Section 4.1. The main input here comes from the work of Ana Rodriguez, Juan L. Vazquez and Juan R. Esteban [RVE97], namely the existence result and an important comparison principle which we exploited in [GT11] to show uniqueness in this class.

Outline

The first Section 4.1 summarises the theory of the *logarithmic fast diffusion equation* which has been mainly studied on the Euclidean $\mathbb{R}^n$; we will explain its relation to the Ricci flow and the *fast diffusion equation*, and its physical origin. We are also going to exploit some of these results in the conformally flat case on the plane. The following section 4.2 is about *a priori* barriers for instantaneously complete Ricci flows, whose key (Lemma 4.2.1) is a combination of Chen's very general *a priori* estimate for a Ricci flow's scalar curvature (Theorem 2.5.2) and a Schwarz Lemma by Shing-Tung Yau (Theorem A.3.1). In the conformally hyperbolic case under the assumption that the solution is initially bounded from above by some (possibly large) multiple of a hyperbolic metric, we are going to bootstrap these barriers ($\to$ §4.3) and obtain smooth convergence of the rescaled flow, curvature estimates and thus long-time existence. No curvature assumptions will be made on the flow at any time, and yet we derive pointwise and also derivative bounds which will be of fundamental importance in the later proofs.

In Section 4.4 we show several comparison principles and combine them with the *a priori* estimates (barriers) to prove uniqueness results. Section 4.5 contains the existence results — in the conformally hyperbolic case based on the preceding *a priori* estimates of Section 4.3 and in the conformally Euclidean case using the corresponding result from the theory of the *logarithmic fast diffusion equation* (§4.1). Finally, we will describe the long-time behaviour on the hyperbolic disc ($\to$ §4.6).

4.1 Logarithmic fast diffusion equation

There is an independent interest and extensive literature on (4.1) which after the change of variables $v = \mathrm{e}^{2u}$ is mostly called the *logarithmic fast diffusion equation*:

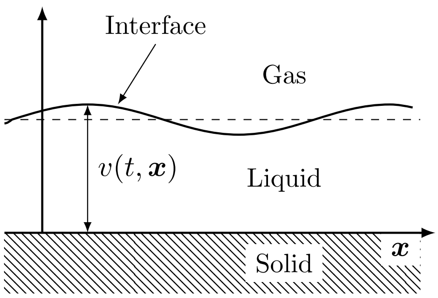
$$\frac{\partial}{\partial t} v = \Delta \log v = \operatorname{div}\left(v^{-1} \operatorname{grad} v\right). \tag{4.2}$$

Virtually all of the literature in this direction considers the equation on $\mathbb{R}^2$ or $\mathbb{R}^n$ for $n > 2$, and we will appeal to some of these results in the conformally Euclidean case. However, King Ming Hui (e.g. [Hui02]) has considered the problem also on bounded domains in $\mathbb{R}^2$. Typically the logarithmic fast diffusion literature considers solutions with some sort of growth condition at infinity, sometimes phrased in terms of membership of an L^p space, and here we are replacing these conditions with the requirement of the associated Ricci flow to be instantaneously complete.

4.1.1 Physical occurence

The logarithmic fast diffusion equation arises in several physical models. Karl E. Lonngren and Akira Hirose derive (4.2) by combining the equation of continuity, Maxwellian electrons and Ohm's law in order to model the expansion of an electron cloud into vacuum [LH76], where v corresponds roughly to the electron density. Further appearances of (4.2) include gas kinetics [McK75] and ion exchange kinetics [HP58].

However, the most influential connection to the logarithmic fast diffusion equation comes from thin film dynamics, in particular in our two-dimensional situation. Malcolm B. Williams and Stephen H. Davis model the height of the interface separating a thin liquid layer on a rigid plate from a passive gas on $v(t, \boldsymbol{x})$. If the layer is sufficiently thin (i.e. $\sim 10 - 100\,\mathrm{nm}$) such that the van der Waals interaction is effective, they deduce in [WD82, Eq. (38a)] the evolution



$$\frac{\partial}{\partial t} v = \mp \Delta \log v - \operatorname{div}\left(v^3 \operatorname{grad}(\Delta v)\right). \tag{4.3}$$

The sign on the right-hand side corresponds to the situations when the van der Waals forces are attractive or repulsive. In the former case small disturbances can grow and lead to a rupture of the film, while in the latter case the uniform film is stable. Further numerical computations [WD82], [BBD88] suggest that the fourth order term in (4.3) which corresponds to the surface tension, can be neglected in some situations, yielding (4.2) along with its backwards sibling (see also [DDD96, §1.5]).

4.1.2 Existence and non-uniqueness

While Lang-Fang Wu gave a proof of existence of the associated Ricci flow under several additional assumptions to the initial metric [Wu93] based on Shi's Theorem 2.4.1, the first general existence result is due to Panagiota Daskalapoulos and Manuel del Pino.

Theorem 4.1.1 (Daskalapoulos and del Pino [DdP95, Theorem 1.2])**.** *For every* $0 \leq v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $\eta > 2$ *there exists a solution* $v \in \mathrm{C}^\infty\big((0, T_\eta) \times \mathbb{R}^2\big) \cap \mathrm{C}\big([0, T_\eta), \mathrm{L}^1(\mathbb{R}^2)\big)$ *of the* logarithmic fast diffusion equation (4.2) *with* $v(0, \cdot) = v_0$*, the maximal time*

$$T_\eta = \frac{1}{2\pi\,\eta} \int_{\mathbb{R}^2} v_0 \,\mathrm{d}\mu \tag{4.4}$$

and the property that

$$\int_{\mathbb{R}^2} v(t) \,\mathrm{d}\mu = \int_{\mathbb{R}^2} v_0 \,\mathrm{d}\mu - 2\pi\eta t. \tag{4.5}$$

In the same article they also show existence of solutions with $\eta = 2$ and possibly infinite mass, i.e. $\|v_0\|_{\mathrm{L}^1(\mathbb{R}^2)} = \infty$ [DdP95, Theorem 1.1]. However, the characterisation of these solutions as maximal solutions (i.e. the associated Ricci flow is maximally stretched) in this case was done by Emmanuele DiBenedetto and David J. Diller [DD96, Theorem 5.1] and in a different way by Rodriguez, Vazquez and Esteban:

Theorem 4.1.2 (Rodriguez, Vazquez and Esteban [RVE97, Theorem 4.1])**.** *For every* $v_0 \in \mathrm{L}^1\big(\mathbb{R}^2\big)$ *with* $v_0 \geq 0$ *there exists a maximal time* $T = \frac{1}{4\pi}\big\|v_0\big\|_{\mathrm{L}^1(\mathbb{R}^2)} < \infty$ *and a classical solution* $0 \leq v \in \mathrm{C}^\infty\big((0, T) \times \mathbb{R}^2\big) \cap \mathrm{C}\big([0, T), \mathrm{L}^1(\mathbb{R}^2)\big)$ *of the* logarithmic fast diffusion equation (4.2) *with* $v(0, \cdot) = v_0$ *satisfying*

$$\big\|v(t, \cdot)\big\|_{\mathrm{L}^1(\mathbb{R}^2)} = \big\|v_0\big\|_{\mathrm{L}^1(\mathbb{R}^2)} - 4\pi t \qquad \text{for all } t \in [0, T). \tag{4.6}$$

Moreover, we have $v(t, \cdot) > 0$ *for all* $t \in (0, T)$ *and* v *satisfies the minimal decay condition, i.e. for some constant* $c > 0$ *we have*

$$v(t, \boldsymbol{x}) \geq c\, \frac{2t}{|\boldsymbol{x}|^2 (\log |\boldsymbol{x}|)^2} \qquad \text{for all } (t, \boldsymbol{x}) \in (0, T) \times \big(\mathbb{R}^2 \setminus \mathbb{B}(\mathbf{0}; 2)\big). \tag{4.7}$$

In the same article they also show that this decay condition (4.7) is sufficient for a solution to be maximal which we will discuss in detail in Section 4.4.1. Note that the associated metric $v(t)|\mathrm{d}z|^2$ for a $v(t)$ satisfying this minimal decay condition (4.7) is complete for all $t \in (0, T)$.

4.1.3 Fast diffusion equation

In the literature, (4.2) is regarded as the (at least) formal limit of the *fast diffusion equation* on $\mathbb{R}^n$

$$\frac{\partial}{\partial t} v = \frac{1}{m} \Delta v^m = \operatorname{div}\left(v^{m-1} \operatorname{grad} v\right) \qquad \text{as } m \searrow 0. \tag{4.8}$$

While the Cauchy problem of (4.8) and $v(0,\cdot) \in \mathrm{L}^1(\mathbb{R}^n)$ always admits a unique solution for all time if $m > 0$, VAZQUEZ showed the non-existince [Vaz92], if either $m \leq -1$ and $n = 1$, or $m < 0$ and $n = 2$, or $m \leq 0$ and $n \geq 3$. Hence, our situation where $n = 2$ and $m = 0$ is the borderline case between unique existence ($m > 0$) and non-existence ($m < 0$), where there still exists a solution but it is no longer unique ($\rightarrow$ Theorem 4.1.1; see also [RV95]); in addition, a loss of mass (4.6) can be observed in contrast to solutions with $m > 0$ which conserve the mass. Also in [RVE97, §6], the authors present an alternative construction of the *maximal* solution from Theorem 4.1.2 as the limit of solutions of the fast diffusion equation (4.8) as $m \searrow 0$.

4.2 Barriers

4.2.1 Barriers for conformally hyperbolic solutions

All estimates and bounds in the next Section 4.3 are based on the following Lemma which provides sharp (hyperbolic) barriers for conformally hyperbolic and instantaneously complete Ricci flows which are initially bounded from above by some multiple of the hyperbolic metric. Moreover, the Lemma shows that the important lower barrier, namely the *big-bang* hyperbolic Ricci flow $\frac{1}{2t} g_{\mathbb{H}}$ is characteristic for any instantaneously complete Ricci flow which is conformally hyperbolic.

Lemma 4.2.1 ([GT11, Lemma 2.1]). *Let $\big(\mathcal{M}^2, g_{\mathbb{H}}\big)$ be a complete hyperbolic surface and let $\big(g(t)\big)_{t\in[0,T]}$ be a Ricci flow on $\mathcal{M}$ which is conformally equivalent to $g_{\mathbb{H}}$.*

(i) If $\big(g(t)\big)_{t\in[0,T]}$ is instantaneously complete, then

$$(2t) g_{\mathbb{H}} \leq g(t) \qquad \textit{for all } t \in (0, T]. \tag{4.9}$$

Moreover, if $\big(g(t)\big)_{t\in[0,T]}$ is complete and $\mathrm{K}_{g(0)} \geq -\kappa_0$ for some $\kappa_0 > 0$, then

$$\big(2t + \kappa_0^{-1}\big) g_{\mathbb{H}} \leq g(t) \qquad \textit{for all } t \in [0, T]. \tag{4.10}$$

(ii) If there exists a constant $M > 0$ such that $g(0) \leq M g_{\mathbb{H}}$, then

$$g(t) \leq \big(2t + M\big) g_{\mathbb{H}} \qquad \textit{for all } t \in [0, T]. \tag{4.11}$$

PROOF. To show the lower barriers (4.9) and (4.10) we use CHEN's *a priori* estimate for the Gaussian curvature to obtain the lower curvature bounds $-\frac{1}{2t} \leq \mathrm{K}_{g(t)}$ for all $t \in (0, T]$ (Corollary 2.5.3), or $-\frac{1}{\kappa_0^{-1} - 2t} \leq \mathrm{K}_{g(t)}$ for all $t \in [0, T]$ if $g(0)$ is complete with curvature bounded from below $\mathrm{K}_{g(0)} \geq -\kappa_0 > -\infty$ (Theorem 2.5.2). In both cases YAU's Schwarz lemma (Corollary A.3.3) allows us then to compare $g(t)$ with $g_{\mathbb{H}}$, establishing (4.9) and (4.10).

(ii) Without loss of generality we may (possibly after lifting to its universal cover) assume $\mathcal{M}^2 = \mathbb{D}$ and write $g(t) = \mathrm{e}^{2u(t)} |\mathrm{d}z|^2$. To prove the upper barrier, consider for small $\delta > 0$, $u\big|_{\overline{\mathbb{D}_{1-\delta}}}$ and write the conformal factor of a complete Ricci flow on the disc of radius $1 - \delta$ with Gaussian curvature initially $-M^{-1}$ as

$$h_\delta(t, z) := \log \frac{2(1-\delta)}{(1-\delta)^2 - |z|^2} + \frac{1}{2} \log\big(2t + M\big).$$

Note that u is continuous on $[0, T] \times \overline{\mathbb{D}_{1-\delta}}$ and $h_\delta(t, z) \geq h_\delta(0, z) \rightarrow \infty$ as $z \rightarrow \partial\mathbb{D}_{1-\delta}$ for all $t \in [0,T]$. Also, with this choice of M, we have initially $u\big|_{\mathbb{D}_{1-\delta}}(0,\cdot) \le h_0\big|_{\mathbb{D}_{1-\delta}}(0,\cdot) \le h_\delta(0,\cdot)$. Therefore the requirements of an elementary comparison principle for the Ricci flow ($\to$ Corollary 2.3.2) are fulfilled, and we may deduce that $u\big|_{\mathbb{D}_{1-\delta}} \le h_\delta$ holds throughout $[0,T] \times \mathbb{D}_{1-\delta}$. Since h_δ is continuous in δ, letting $\delta \searrow 0$ yields (4.11). $\square$

4.2.2 Hyperbolic barriers for the whole plane

Whilst Lemma 4.2.1(i) gives good lower barriers for (instantaneously) complete solutions on a hyperbolic disc $\mathbb{D} \subset \mathbb{C}$, it cannot be used directly on the whole plane $\mathbb{C}$, because it does not admit a hyperbolic metric. However, by considering the plane with a disc taken off which does admit a hyperbolic metric, we can still exploit YAU's Schwarz lemma (Theorem A.3.1).

Lemma 4.2.2. *For some $\kappa \in (0,\infty)$ let $\mathrm{e}^{2u}|\mathrm{d}z|^2$ be a complete metric on the plane $\mathbb{C}$ with curvature bounded from below $\mathrm{K}_u \ge -\kappa$. Then there exists a constant $C = C\left(\|u\|_{\mathrm{C}^2(\mathbb{D}_2)}\right) > 0$ such that for all $z \in \mathbb{C} \setminus \mathbb{D}_2$*

$$u(z) \ge -\log\big(|z|\log|z|\big) - \frac{1}{2}\log\max\big\{\kappa, C\big\}. \tag{4.12}$$

PROOF. Pick any cut-off function $\varphi \in \mathrm{C}_c^\infty\big(\mathbb{D}_2,[0,1]\big)$ with $\varphi \equiv 1$ in $\mathbb{D}_{3/2}$. Furthermore, let $\mathrm{e}^{2h}|\mathrm{d}z|^2$ be the complete hyperbolic metric on $\mathbb{C} \setminus \overline{\mathbb{D}}$, i.e. $h(z) = -\log\big(|z|\log|z|\big)$. Finally, consider an interpolated metric defined by the conformal factor $v \in \mathrm{C}^\infty\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big)$ given by

$$v(z) := \varphi(z)\cdot h(z) + \big(1-\varphi(z)\big)\cdot u(z) \qquad \text{for all } z \in \mathbb{C}\setminus\overline{\mathbb{D}}.$$

Thus we can estimate the Gaussian curvature of $\mathrm{e}^{2v}|\mathrm{d}z|^2$ by

$$\mathrm{K}_v(z) \ge \begin{Bmatrix} -1 & \text{for } |z| \le \frac{3}{2}, \text{ i.e. where } v = h \\ -C & \text{for } \frac{3}{2} < |z| < 2 \\ -\kappa & \text{for } |z| \ge 2, \text{ i.e. where } v = u \end{Bmatrix} \ge -\max\big\{\kappa, C\big\}$$

with $C = C\left(\|u\|_{\mathrm{C}^2(\mathbb{D}_2)}\right) \in (0,\infty)$. Comparing $\mathrm{e}^{2v}|\mathrm{d}z|^2$ with $\mathrm{e}^{2h}|\mathrm{d}z|^2$ using YAU's generalised Schwarz lemma (Corollary A.3.3) yields for all $z \in \mathbb{C}\setminus\mathbb{D}_2$

$$u(z) = v(z) \ge h(z) - \frac{1}{2}\log\max\big\{\kappa, C\big\}. \qquad \square$$

As in the proof of Lemma 4.2.1(i) we can now exploit the lower bound to the curvature of a Ricci flow provided by CHEN's curvature estimate (Corollary 2.5.3).

Corollary 4.2.3 ([GT11, Theorem 3.2])**.** *Let $\left(\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\right)_{t\in[0,T]}$ be an instantaneoulsy complete Ricci flow on the plane $\mathbb{C}$. Then there exists a constant $C < \infty$ depending only on T and $\sup_{[0,T]}\|u(t)\|_{\mathrm{C}^2(\mathbb{D}_2)}$ such that for all $z \in \mathbb{C}\setminus\mathbb{D}_2$ and $t \in (0,T]$*

$$u(t,z) \ge -C - \log\big(|z|\log|z|\big) + \frac{1}{2}\log(2t). \tag{4.13}$$

Note, that this decay condition (4.13) of instantaneously complete Ricci flows on the plane originates from the decay of the hyperbolic cusp metric. Then again, RODRIGUEZ, VAZQUEZ and ESTEBAN found the very same decay (4.7) for the *logarithmic fast diffusion equation* by direct analytical methods. It enabled them to prove a very strong comparison principle ($\to$ Theorem 4.4.1).

Proof. By Corollary 2.5.3 we have the curvature estimate $\mathrm{K}_{g(t)} \geq -\frac{1}{2t}$ for all $t \in (0,T]$. Choosing $C := \frac{1}{2}\sup_{t\in(0,T]}\big[\log\big(2t\,\tilde{C}(t)\big)\big]_+$, where $\tilde{C}(t) > 0$ is the constant depending on $\big\|u(t)\big\|_{\mathrm{C}^2(\mathbb{D}_2)}$ provided by Lemma 4.2.2, we estimate

$$\begin{aligned} u(t,z) + \log\big(|z|\log|z|\big) &\geq -\frac{1}{2}\log\max\left\{\frac{1}{2t}, \tilde{C}(t)\right\} \\ &= -\frac{1}{2}\log\max\big\{1, 2t\,\tilde{C}(t)\big\} + \frac{1}{2}\log(2t) \\ &\geq -C + \frac{1}{2}\log(2t) \qquad \text{for all } (t,z) \in (0,T] \times \big(\mathbb{C}\setminus\mathbb{D}_2\big). \quad \square \end{aligned}$$

4.3 *A priori* estimates for conformally hyperbolic flows

4.3.1 Normalised Ricci flow

Given a Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ on a conformally hyperbolic surface $\mathcal{M}^2$ we choose $\psi(t) = \frac{1}{2t}$ and $t_0 = \frac{1}{2}$ in Proposition 2.2.3, to obtain $\bar{\kappa} = -1$ and

$$\frac{\partial}{\partial \bar{t}}\bar{g}(\bar{t}) = -2(\mathrm{K}_{\bar{g}(\bar{t})} + 1)\,\bar{g}(\bar{t}), \tag{4.14}$$

Consequently, we have the conversions between normalised and unnormalised Ricci flow

$$\begin{aligned} \bar{g}\big(\bar{t}(t)\big) &= \frac{1}{2t}g(t) \qquad \text{and} \qquad \bar{t}(t) = \frac{1}{2}\log(2t) \qquad \text{for all } t \in (0,T), \\ g\big(t(\bar{t})\big) &= \mathrm{e}^{-2\bar{t}}\bar{g}(\bar{t}) \qquad \text{and} \qquad t(\bar{t}) = \frac{1}{2}\mathrm{e}^{2\bar{t}} \qquad \text{for all } \bar{t} \in \big(-\infty,\bar{T}\big), \end{aligned} \tag{4.15}$$

with $\bar{T} := \frac{1}{2}\log(2T)$. Note, that the time $t = 0$ of the unnormalised Ricci flow does not have a counterpart of the normalised variant. Moreover, the latter one is ancient. If we write locally $\mathrm{e}^{2\bar{u}(\bar{t})}|\mathrm{d}z|^2 = \bar{g}(\bar{t})$, the evolution (4.14) can be rephrased by

$$\frac{\partial}{\partial \bar{t}}\bar{u} = \mathrm{e}^{-2\bar{u}}\Delta\bar{u} - 1 = \operatorname{div}\big(\mathrm{e}^{-2\bar{u}}\,\mathrm{D}\bar{u}\big) + 2\mathrm{e}^{-2\bar{u}}\,\big|\mathrm{D}\bar{u}\big|^2 - 1. \tag{4.16}$$

4.3.2 C^k bounds

In this section we bootstrap the barriers of the previous section to obtain estimates for higher derivatives. In the following lemma we start with a local bound around the origin by switching to the normalised flow (4.16) where the necessary C^0-bounds can be established away from $t = 0$ independently of the (maximal) time.

Lemma 4.3.1. *Let $\big(\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\big)_{t\in[0,T]}$ be an instantaneously complete Ricci flow on the unit disc $\mathbb{D}$ and for $r \in (0,1]$ let $\mathrm{e}^{2h_r}|\mathrm{d}z|^2$ be the complete hyperbolic metric on the disc $\mathbb{D}_r$ of radius r. If $u(0)\big|_{\mathbb{D}_r} \leq h_r + \frac{1}{2}\log M$ for some $M > 0$, then there exists for any $\delta \in \big(0, \min\{T, r\}\big)$ and $k \in \mathbb{N}_0$ a constant $C_k = C_k(k, r, \delta, M) < \infty$ such that for all $t \in [\delta, T]$*

$$\left\|u(t,\cdot) - \frac{1}{2}\log(2t)\right\|_{\mathrm{C}^k(\mathbb{D}_{r-\delta})} \leq C_k. \tag{4.17}$$

Proof. Lemma 4.2.1 provides the following upper and lower bounds for u,

$$\log\frac{2}{1-|z|^2} + \frac{1}{2}\log(2t) \leq u(t,z) \leq \log\frac{2r}{r^2-|z|^2} + \frac{1}{2}\log\big(2t + M\big) \tag{4.18}$$

for all $t \in (0,T]$ and $z \in \mathbb{D}_r$. $\lvert u \rvert$ cannot be bounded uniformly away from $t = 0$ independently of T, but using the conversion (4.15) we may rewrite (4.18) in terms of the normalised flow $\bar{u}(\bar{t})$

$$\log \frac{2}{1-|z|^2} \le \bar{u}(\bar{t}, z) \le \log \frac{2r}{r^2 - |z|^2} + \frac{1}{2} \log \left(1 + M\mathrm{e}^{-2\bar{t}}\right), \tag{4.19}$$

for $z \in \mathbb{D}_r$, where we have uniform bounds which are independent of $T =: \frac{1}{2}\mathrm{e}^{2\bar{T}}$ if $\bar{t}$ is bounded from below. Indeed, fixing $\delta \in \big(0, \min\{T, r\}\big)$ and hence $\bar{\delta} := \frac{1}{2}\log(2\delta)$ there exists a constant $C = C(M, \delta, r) > 0$ such that

$$\sup_{[\bar{\delta}-1, \bar{T}] \times \mathbb{D}_{r-\delta/2}} \lvert \bar{u} \rvert \le C < \infty.$$

Thus the evolution equation (4.16) for $\bar{u}(\bar{t}, z)$ is uniformly parabolic on $[\bar{\delta}-1, \bar{T}] \times \mathbb{D}_{r-\delta/2}$, which allows us to apply standard parabolic theory, i.e. Theorem B.1.1 to establish Hölder bounds on $\bar{u}$ and then use parabolic Schauder estimates (Corollary B.2.2) to bootstrap and obtain, for any $k \in \mathbb{N}_0$, constants $C_k = C_k(k, r, \delta, C) > 0$ such that

$$\| \, \bar{u} \, \|_{\mathrm{C}^{k+\frac{\alpha}{2}, 2k+\alpha}\left([\bar{\delta}, \bar{T}] \times \mathbb{D}_{r-\delta}\right)} \le C_k$$

yielding (4.17) after converting back to the unnormalised flow by virtue of (4.15). $\square$

The next theorem extends these local estimates to global ones which will also show that in this setting the rescaled solution converges smoothly and uniformly to the hyperbolic metric as time goes to infinity.

Theorem 4.3.2 ([GT11, Theorem 2.3])**.** *Let $(\mathcal{M}^2, g_{\mathbb{H}})$ be a complete hyperbolic surface, and suppose $\big(g(t)\big)_{t \in [0,T]}$ is an instantaneously complete Ricci flow on $\mathcal{M}$ which is conformally equivalent to $g_{\mathbb{H}}$. If $g(0) \le M g_{\mathbb{H}}$ for some constant $M > 0$, then for all $k \in \mathbb{N}_0$ and any $\eta \in (0,1)$ and $\delta \in (0,T)$ (however small) there exists a constant $C = C(k, \eta, \delta, M) < \infty$ such that for all $t \in [\delta, T]$ there holds*

$$\left\| \frac{1}{2t} g(t) - g_{\mathbb{H}} \right\|_{\mathrm{C}^k(\mathcal{M}, g_{\mathbb{H}})} \le \frac{C}{t^{1-\eta}}. \tag{4.20}$$

Proof. Fix any point $p \in \mathcal{M}$, and let $\pi : \mathbb{D} \to \mathcal{M}$ be a universal covering of $\mathcal{M}$ with $\pi(0) = p$. Without loss of generality we may then write the lifted metrics $\pi^* g(t) = \mathrm{e}^{2u(t)} |\mathrm{d}z|^2$ and $\pi^* g_{\mathbb{H}} = \mathrm{e}^{2h} |\mathrm{d}z|^2$, the latter being the complete hyperbolic metric on the disc. We also have $\mathrm{e}^{2u(0)} \le M \mathrm{e}^{2h}$ by hypothesis.

Using Lemma 4.3.1 we obtain for every $k \in \mathbb{N}$ constants $C'_k = C'_k(k, \delta, M) > 0$ such that we have uniform C^k-bounds for all $t \in [\delta, T]$

$$\sup_{\mathbb{D}_{1/2}} \left| \mathrm{D}^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right| = \sup_{\mathbb{D}_{1/2}} \left| \mathrm{D}^k \left(\mathrm{e}^{2\left(u(t) - \frac{1}{2}\log(2t)\right)} \mathrm{e}^{-2h} \right) \right| \le C'_k. \tag{4.21}$$

By virtue of Lemma 4.2.1 there is a much stronger C^0-estimate for all $t \in (0,T]$ on $\mathbb{D}$

$$0 \le \frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \le \frac{M}{2t}. \tag{4.22}$$

Now fix $\eta \in (0,1)$ and combine (4.22) and (4.21) with the interpolation inequality of

Lemma B.3.1 to obtain for every $k \in \mathbb{N}$ constants $C''_k = C''_k(k, \eta, \delta, M) > 0$ and $l = \lceil k/\eta \rceil$ such that for all $t \in [\delta, T]$

$$\left| \mathrm{D}^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right| (0) \leq C''_k t^{-(1-\eta)}. \tag{4.23}$$

Note that the case $k = 0$ of (4.23) is already dealt with by (4.22). Then we estimate using Lemma B.3.2 (with constant $c = c(k) > 0$) and (4.23) for all $t \in [\delta, T]$

$$\begin{aligned} \left| {}^{g_{\mathbb{H}}}\nabla^k \left(\frac{1}{2t} g(t) - g_{\mathbb{H}} \right) \right|_{g_{\mathbb{H}}} (p) &= \left| {}^{\pi^* g_{\mathbb{H}}}\nabla^k \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \pi^* g_{\mathbb{H}} \right|_{\pi^* g_{\mathbb{H}}} (0) \\ &\leq c \sum_{j=0}^{k} \left| \mathrm{D}^j \left(\frac{1}{2t} \mathrm{e}^{2(u(t)-h)} - 1 \right) \right|_{|\mathrm{d}z|^2} (0) \\ &\leq c \sum_{j=0}^{k} C''_j t^{-(1-\eta)} =: C'''_k t^{-(1-\eta)}. \end{aligned} \tag{4.24}$$

Since $p \in \mathcal{M}$ was chosen arbitrarily and the constants C'''_k are independent of p (and of π), (4.24) holds for all $p \in \mathcal{M}$ and we conclude that with $C = C(k, \eta, \delta, M) < \infty$,

$$\left\| \frac{1}{2t} g(t) - g_{\mathbb{H}} \right\|_{\mathrm{C}^k(\mathcal{M}, g_{\mathbb{H}})} \leq C\, t^{-(1-\eta)}$$

for all $t \in [\delta, T]$. □

4.3.3 Curvature bounds and long-time existence

The uniform C^2-bounds of Theorem 4.3.2 yield uniform curvature bounds away from $t = 0$ for instantaneously complete Ricci flows which are initially bounded from above by a multiple of the hyperbolic metric. This allows us to extend such solutions to be immortal.

Proposition 4.3.3 (Curvature decay. [GT11, Proposition 2.4]). *Let $(\mathcal{M}^2, g_{\mathbb{H}})$ be a complete hyperbolic surface and $(g(t))_{t \in [0,T]}$ an instantaneously complete Ricci flow on $\mathcal{M}$, conformally equivalent to $g_{\mathbb{H}}$. If $g(0) \leq M g_{\mathbb{H}}$ for some constant $M > 0$, then for all $\delta \in (0, T]$ there exists a constant $C = C(M, \delta) < \infty$ such that for all $t \in [\delta, T]$ we have*

$$\left| \mathrm{K}_{g(t)} \right| \leq \frac{C}{t}.$$

Proof. By Theorem 4.3.2 there exists a constant $C' = C'(\delta, M) > 0$ such that for all $t \in [\delta, T]$ we have

$$\left| \mathrm{K}_{g(t)} \right| = \frac{1}{2t} \left| \mathrm{K}_{\frac{1}{2t} g(t)} \right| \leq \frac{C'}{2t}. \qquad \square$$

Note that from Theorem 4.3.2 we even have

$$2t\, \mathrm{K}_{g(t)} = \mathrm{K}_{\frac{1}{2t} g(t)} \longrightarrow -1 \quad \text{uniformly as } t \to \infty.$$

Corollary 4.3.4 (Long-time existence. [GT11, Corollary 2.6]). *Let $(\mathcal{M}^2, g_{\mathbb{H}})$ be a complete hyperbolic surface, and $(g(t))_{t \in [0,T]}$ an instantaneously complete Ricci flow on $\mathcal{M}$, conformally equivalent to $g_{\mathbb{H}}$. If $g(0) \leq M g_{\mathbb{H}}$ for some constant $M > 0$, then* *there exists a unique instantaneously complete extension of $g(t)$ defined for all time $t \in [0, \infty)$.*

Moreover, if for some $\kappa_0 < \infty$ there holds $\mathrm{K}_{g(t)} \leq \kappa_0$ for all $t \in [0, T]$, then there exists a constant $\kappa = \kappa(M, T, \kappa_0) < \infty$ such that

$$\mathrm{K}_{g(t)} \leq \kappa \qquad \textit{for all } t \in [0, \infty). \tag{4.25}$$

Proof. The long-time existence is a direct consequence of Corollary 2.4.2 and the *a priori* curvature bounds of Proposition 4.3.3: We have for all $t \in \left[{}^{T}\!/2, T\right]$

$$\left|\mathrm{K}_{g(t)}\right| \leq \frac{C}{t} \leq \frac{2C}{T} =: \kappa_1 < \infty. \tag{4.26}$$

By Corollary 2.4.2 there exist a $\tau > 0$ depending only on κ_1 and a unique extension $\left(\hat{g}(t)\right)_{t \in [0, T+\tau]}$ with the very same curvature bound κ_1 in (4.26) on the larger time interval $\left[{}^{T}\!/2, T + \tau\right]$. Iterating these arguments we obtain for every $j \in \mathbb{N}$ an extension $\left(\hat{g}(t)\right)_{t \in [0, T+j\tau]}$ with bounded curvature $\left|\mathrm{K}_{\hat{g}(t)}\right| \leq \kappa_1$ for all $t \in \left[{}^{T}\!/2, T + j\tau\right]$, and therefore we can continue $g(t)$ to exist for all time $t \in [0, \infty)$. Since the curvature is bounded away from $t = 0$, by Corollary 2.4.2 the extension is also unique among other instantaneously complete extensions.

In order to show the uniform upper bound for the curvature in (4.25), note that (4.26) is true for all $t \in \left[{}^{T}\!/2, \infty\right)$. Combining that bound with κ_0 for times $t \in [0, {}^{T}\!/2]$, we conclude the theorem with $\kappa := \max\{\kappa_0, \kappa_1\}$. $\square$

4.4 Uniqueness

4.4.1 Conformally Euclidean solutions on the plane $\mathbb{C}$

The proof of uniqueness of instantaneously complete Ricci flows on the plane $\mathbb{C}$ is based on the following comparison principle for the *logarithmic fast diffusion equation* by Rodriguez, Vazquez and Esteban. Note that the mandatory decay condition (4.27) for the conformal factor obviously implies the associated Ricci flow to be instantaneously complete. On the other hand, by Corollary 4.2.3 the converse is also true, so the conformal factor of any instantaneously complete Ricci flow satisfies that decay condition (4.27).

Theorem 4.4.1 (Rodriguez, Vazquez, Esteban. Variant of [RVE97, Corollary 2.3])**.** *Let $\left(\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\right)_{t \in [0,T]}$ and $\left(\mathrm{e}^{2v(t)}|\mathrm{d}z|^2\right)_{t \in [0,T]}$ be two Ricci flows on the plane $\mathbb{C}$. If there exists a constant $C < \infty$ such that $u(t)$ satisfies the decay condition*

$$u(t, z) \geq -C - \log\bigl(|z| \log |z|\bigr) + \frac{1}{2}\log(2t) \quad \textit{for all } (t, z) \in (0, T) \times (\mathbb{C} \setminus \mathbb{D}_2) \tag{4.27}$$

and $u(0) \geq v(0)$, then $u(t) \geq v(t)$ for all $t \in [0, T]$.

Proof. For some $R \gg 1$ consider the smooth cut-off function $\zeta_R \in \mathrm{C}_c^\infty\bigl(\mathbb{C}, [0, 1]\bigr)$ such that $\zeta_R\big|_{\mathbb{D}_R} \equiv 1$ and $\operatorname{spt} \zeta_R \subset \mathbb{D}_{R^2}$ which is to be determined in more detail later. Abbreviating the time-dependent open subset $\mathcal{U}^+(t) := \{z \in \mathbb{C} : (v - u)(t, z) > 0\}$, by Lemma B.3.3 we may differentiate at almost each time $t \in (0, T)$ and estimate for any

$m \in (0,1)$

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{C}}\left[\mathrm{e}^{2v}-\mathrm{e}^{2u}\right]_+\zeta_R\,\mathrm{d}\mu &= \int_{\mathcal{U}^+}\frac{\partial}{\partial t}\left(\mathrm{e}^{2v}-\mathrm{e}^{2u}\right)\zeta_R\,\mathrm{d}\mu = \int_{\mathcal{U}^+}2\Delta(v-u)\,\zeta_R\,\mathrm{d}\mu\\
&= \int_{\mathbb{C}}2[v-u]_+\Delta\zeta_R\,\mathrm{d}\mu + 2\int_{\partial\mathcal{U}^+}\left\langle -\frac{\mathrm{D}(v-u)}{|\mathrm{D}(v-u)|},\zeta_R\,\mathrm{D}(v-u)-(v-u)\,\mathrm{D}\zeta_R\right\rangle \mathrm{d}\mathcal{H}^1\\
&\leq \int_{\mathbb{C}}\frac{\mathrm{e}^{-2mu}}{m}\left[\mathrm{e}^{2v}-\mathrm{e}^{2u}\right]_+^m\left|\Delta\zeta_R\right|\mathrm{d}\mu - 2\int_{\partial\mathcal{U}^+}\left|\mathrm{D}(v-u)\right|\zeta_R\,\mathrm{d}\mathcal{H}^1\\
&\leq \frac{1}{m}\int_{\mathbb{C}}\mathrm{e}^{-2mu}\zeta_R^{-m}\left|\Delta\zeta_R\right|\Big(\left[\mathrm{e}^{2v}-\mathrm{e}^{2u}\right]_+\zeta_R\Big)^m\mathrm{d}\mu\\
&\leq \frac{1}{m}\underbrace{\left(\int_{\mathbb{C}}\mathrm{e}^{-\frac{2m}{1-m}u}\zeta_R^{-\frac{m}{1-m}}\left|\Delta\zeta_R\right|^{\frac{1}{1-m}}\mathrm{d}\mu\right)^{1-m}}_{=:I(t)}\cdot\left(\int_{\mathbb{C}}\left[\mathrm{e}^{2v}-\mathrm{e}^{2u}\right]_+\zeta_R\,\mathrm{d}\mu\right)^m
\end{aligned}
\tag{4.28}
$$

using the inequality $[x]_+ \leq \frac{1}{m}\left[\mathrm{e}^x-1\right]_+^m$ for any $x\in\mathbb{R}$ and the Hölder inequality. In order to estimate $I(t)$, we refine the cut-off function

$$
\zeta_R(z) := \left(Z\left(\frac{\log|z|}{\log R}\right)\right)^k
$$

where $k \geq 2$ and $0 \leq Z \in \mathrm{C}^\infty(\mathbb{R})$ is a nonnegative function such that $Z(\varsigma)=1$ for $\varsigma \leq 1$ and $Z(\varsigma)=0$ for $\varsigma \geq 2$, and calculate its Laplacian

$$
\Delta\zeta_R(z) = \frac{kZ^{k-2}}{|z|^2(\log R)^2}\left((k-1)Z'^2+ZZ''\right).
$$

Choosing $k=\frac{2}{1-m}$ and abbreviating $\tilde{Z}_m=(k-1)Z'^2+ZZ''$ we estimate using the decay for u (4.27)

$$
\begin{aligned}
I(t)^{1-m} &= \left(\int_{\mathbb{C}}\mathrm{e}^{-\frac{2m}{1-m}u}\zeta_R^{-\frac{m}{1-m}}\left|\Delta\zeta_R\right|^{\frac{1}{1-m}}\mathrm{d}\mu\right)^{1-m}\\
&= \frac{2}{(1-m)(\log R)^2}\left(\int_{\mathbb{D}_{R^2}\setminus\mathbb{D}_R}\mathrm{e}^{-\frac{2m}{1-m}u}|z|^{-\frac{2}{1-m}}\left|\tilde{Z}_m\right|^{\frac{1}{1-m}}\mathrm{d}\mu\right)^{1-m}\\
&\leq \frac{2\mathrm{e}^{2mC}t^{-m}}{(1-m)(\log R)^2}\left(\int_{\mathbb{D}_{R^2}\setminus\mathbb{D}_R}|z|^{-2}\big(\log|z|\big)^{\frac{2m}{1-m}}\left|\tilde{Z}_m\right|^{\frac{1}{1-m}}\mathrm{d}\mu\right)^{1-m}\\
&= \frac{2(2\pi)^{1-m}\mathrm{e}^{2mC}t^{-m}}{(1-m)(\log R)^{1-m}}\left(\int_1^2\varsigma^{\frac{2m}{1-m}}\left|\tilde{Z}_m(\varsigma)\right|^{\frac{1}{1-m}}\mathrm{d}\varsigma\right)^{1-m}\\
&= \tilde{C}(m,C)\frac{t^{-m}}{(\log R)^{1-m}},
\end{aligned}
\tag{4.29}
$$

where we used the change of coordinates $z=R^\varsigma\mathrm{e}^{\mathrm{i}\theta}$ and hence $\mathrm{d}\mu=|z|^2\log R\,\mathrm{d}\varsigma\mathrm{d}\theta$. Assume there exists a time $t_1\in(0,T]$ with $u(t_1,z_1)<v(t_1,z_1)$ for some $z_1\in\mathbb{C}$. Then define

$$
t_0 := \inf\Big\{\tau\in[0,t_1)\,:\,\forall t\in[\tau,t_1]\ \exists z\in\mathbb{C}\text{ such that }u(t,z)<v(t,z)\Big\}.
$$

Hence, we have $u(t_0)\geq v(t_0)$ and for every $t\in(t_0,t_1]$ there exists $R_t\gg 1$ such that

$\int_{\mathbb{C}} \left[e^{2v(t)} - e^{2u(t)}\right]_+ \zeta_R \, d\mu > 0$ for all $R \geq R_t$. Hence, we may integrate (4.28) with respect to $t \in [\tau, t_1]$ for some $\tau \in (t_0, t_1)$

$$\begin{aligned} &\left(\int_{\mathbb{D}_R} \left[e^{2v(t_1)} - e^{2u(t_1)} \right]_+ d\mu \right)^{1-m} \\ &\qquad \leq \left(\int_{\mathbb{D}_{R^2}} \left[e^{2v(\tau)} - e^{2u(\tau)} \right]_+ d\mu \right)^{1-m} + \tilde{C}(m, C) \, \frac{t_1^{1-m} - \tau^{1-m}}{(\log R)^{1-m}} \\ &\overset{\substack{R\to\infty \\ \tau\searrow t_0 \\ m\searrow 0}}{\Longrightarrow} \quad 0 < \int_{\mathbb{C}} \left[e^{2v(t_1)} - e^{2u(t_1)} \right]_+ d\mu \leq \int_{\mathbb{C}} \left[e^{2v(t_0)} - e^{2u(t_0)} \right]_+ d\mu = 0. \quad (\lightning) \end{aligned}$$

$\square$

Corollary 4.4.2. *Every conformally Euclidean, instantaneously complete Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on $\mathbb{C}$ is maximally stretched.*

Proof. By Corollary 4.2.3 the conformal factor $u(t)$ of $g(t) = e^{2u(t)}|dz|^2$ satisfies the decay condition (4.27) of the comparison principle Theorem 4.4.1, and the corollary's statement follows. $\square$

4.4.2 Conformally hyperbolic solutions

The uniqueness of instantaneously complete Ricci flows on conformally hyperbolic surfaces in full generality is still only conjectured. In contrast to the other (conformal) cases, we have even *strong uniqueness* only when we impose additional conditions on the initial surface. Chen introduces this term [Che09] if apart from completeness (for all time) any additional requirements on a solution are only imposed at the initial time. He showed uniqueness in the class of complete flows if the inital surface is non-collapsed and has bounded curvature ($\rightarrow$ Theorem 5.2.7).

Unfortunately, we do not have a comparison result as strong as in the above conformally flat case on $\mathbb{C}$, but we can exploit the weak maximum principle Theorem 2.3.3 for non-compact, complete Riemannian manifolds with bounded curvature to obtain a weaker result.

Theorem 4.4.3 (Geometric comparison principle. Variant of [GT10, Theorem 4.2])**.** *For some $T > 0$ let $\big(g_1(t)\big)_{t\in[0,T]}$ and $\big(g_2(t)\big)_{t\in[0,T]}$ be two conformally equivalent Ricci flows on a complete surface $\big(\mathcal{M}^2, \tilde{g}\big)$ of bounded curvature, and define $Q : [0,T] \times \mathcal{M} \to \mathbb{R}$ to be the function for which $g_1(t) = e^{2Q(t)} g_2(t)$. Suppose further that $g_2(t) \geq \delta\tilde{g}$ for all $t \in [0,T]$, $Q(t)$ satisfies the decay condition (2.18) and that for some non-negative, integrable function $0 \leq f \in L^1\big([0,T]\big)$ we have*

$$e^{2Q(t)} K_{g_1(t)} \leq f(t) \tag{4.30}$$

for all $t \in (0,T]$ on $\mathcal{M}$. If $g_1(0) \leq g_2(0)$, then $g_1(t) \leq g_2(t)$ for all $t \in [0,T]$.

Proof. Differentiating $g_1(t) = e^{2Q(t)} g_2(t)$ yields

$$2\big(-K_{g_1}\big) g_1 = \frac{\partial}{\partial t} g_1 = 2 \frac{\partial}{\partial t} Q \, e^{2Q} g_2 - 2K_{g_2} e^{2Q} g_2 = 2 \left(\frac{\partial}{\partial t} Q - K_{g_2} \right) g_1. \tag{4.31}$$

Using the relation $K_{g_1} = e^{-2Q}\big(-\Delta_{g_2} Q + K_{g_2}\big)$ from Lemma 2.2.1, we obtain where

$Q > 0$

$$\left(\frac{\partial}{\partial t} - \Delta_{g_2}\right) Q = \mathrm{K}_{g_2} - \mathrm{K}_{g_1} - \Delta_{g_2} Q = \left(\mathrm{e}^{2Q} - 1\right) \mathrm{K}_{g_1}$$
$$= \frac{\mathrm{e}^{2Q} - 1}{Q - 0} \mathrm{K}_{g_1} Q = 2\mathrm{e}^{2\xi}\, \mathrm{K}_{g_1} Q \leq 2f\, Q$$

where at each point, ξ was chosen between 0 and Q according to the mean value theorem. Applying the weak maximum principle Theorem 2.3.3 to Q, keeping in mind that $Q(0,\cdot) \leq 0$, we conclude that $Q \leq 0$ throughout $[0,T] \times \mathcal{M}$ as desired. □

In the following theorem which is a slight improvement of [GT11, Theorem 1.7] and [GT10, Theorem 4.1], we require the solution to be bounded initially from above by some multiple of the hyperbolic metric and its curvature to be bounded from above by some integrable time-dependent function for a very short time.

Theorem 4.4.4. *Let $\left(\mathcal{M}^2, g_{\mathbb{H}}\right)$ be a complete hyperbolic surface, and for some constants $T > 0$, $M > 0$, $\varepsilon \in (0,T]$ and some integrable $f \in \mathrm{L}^1\left([0,\varepsilon]\right)$, suppose that $\left(g_1(t)\right)_{t\in[0,T]}$ and $\left(g_2(t)\right)_{t\in[0,T]}$ are two instantaneously complete Ricci flows on $\mathcal{M}$ which are conformally equivalent to $g_{\mathbb{H}}$, with $g_1(0) = g_2(0)$, $g_i(0) \leq M g_{\mathbb{H}}$ and $\mathrm{K}_{g_i(t)} \leq f(t)$ for all $t \in (0,\varepsilon)$, $i \in \{1,2\}$. Then $g_1(t) = g_2(t)$ for all $t \in [0,T]$.*

We cannot apply the comparison principle (Theorem 4.4.3) directly to our instantaneously complete Ricci flow, because one solution is also required to be complete initially and to have a suitable lower barrier. Therefore, we translate and dilate it in time, to achieve exactly this requirement by virtue of the barrier from Lemma 4.2.1(i).

Proof. Without loss of generality we assume that f is a non-negative, integrable function on the whole time-interval, i.e. $0 \leq f \in \mathrm{L}^1\left([0,T]\right)$ such that $\mathrm{K}_{g_i(t)} \leq f(t)$ for all $t \in (0,T]$, $i \in \{1,2\}$ (using Proposition 4.3.3 to extend this curvature bound if necessary). Writing $F(t) := \mathrm{e}^{2\int_0^t f(\tau)\mathrm{d}\tau}$ for all $t \in [0,T]$, we define for small $\delta \in (0, \min\{T, {}^{M}\!/_{2}\})$ the slight adjustment of $g_2(t)$

$$g_{2,\delta}(t) := F(\delta)\, g_2\left(\frac{1}{F(\delta)}(t+\delta)\right) \qquad \text{for all } t \in [0, T-\delta]$$

which is again a Ricci flow:

$$\frac{\partial}{\partial t} g_{2,\delta}(t) + 2\mathrm{Rc}_{g_{2,\delta}(t)} = \left.\frac{\partial}{\partial \tilde{t}} g_2(\tilde{t}) + 2\mathrm{Rc}_{g_2(\tilde{t})}\right|_{\tilde{t} = \frac{1}{F(\delta)}(t+\delta)} = 0.$$

We are now going to apply the comparison principle Theorem 4.4.3 to $g_1(t)$ and $g_{2,\delta}(t)$: First note that for all $t \in [0, T-\delta]$, $g_{2,\delta}(t)$ is complete and has a lower barrier by Lemma 4.2.1 applied to $\left(g_2(t)\right)_{t\in[0,T]}$

$$g_{2,\delta}(t) = F(\delta)\, g_2\left(\frac{1}{F(\delta)}(t+\delta)\right) \geq 2(t+\delta) g_{\mathbb{H}} \geq 2\delta\, g_{\mathbb{H}}. \tag{4.32}$$

Define $Q_\delta \in \mathrm{C}^\infty\left([0,T-\delta] \times \mathcal{M}\right)$ to be the function for which $g_1(t) = \mathrm{e}^{2Q_\delta(t)} g_{2,\delta}(t)$. Combining (4.32) with the upper barrier (4.11) for $g_1(t)$ by Lemma 4.2.1, we obtain for all $t \in [0, T-\delta]$

$$0 = g_1(t) - \mathrm{e}^{2Q_\delta(t)} g_{2,\delta}(t) \le \Big((2t+M) - \mathrm{e}^{2Q_\delta(t)}(2t+2\delta) \Big)\, g_{\mathbb{H}}.$$

Consequently, Q_δ satisfies the decay condition (2.18) required by Theorem 4.4.3, in particular we have $Q_\delta(t) \le \frac{1}{2}\log\frac{2t+M}{2t+2\delta} \le \frac{1}{2}\log\frac{M}{2\delta}$, and also (4.30):

$$\mathrm{e}^{2Q_\delta(t)} \mathrm{K}_{g_1(t)} \le \frac{M}{2\delta} f(t) \qquad \text{for all } t \in [0, T-\delta].$$

Finally, using the upper curvature bound and integrating the Ricci flow equation (2.3) for $g_2(t)$ i.e. $\frac{\partial}{\partial t} g_2(t) \ge -2f(t)\cdot g_2(t)$, we obtain $g_2(t) \ge \frac{1}{F(t)}\, g_2(0)$ for all $t \in [0,T]$, and hence

$$g_{2,\delta}(0) = F(\delta)\, g_2\!\left(\frac{\delta}{F(\delta)}\right) \ge \frac{F(\delta)}{F\!\left(\frac{\delta}{F(\delta)}\right)}\, g_2(0) \ge g_2(0) \ge g_1(0),$$

since $F'(t) = 2f(t)\cdot F(t) \ge 0$ and $F(t) \ge 1$ for all $t \in [0,T]$. Therefore, for all $\delta \in (0, \min\{T, {}^{M}\!/{}_{2}\})$ we may apply the comparison principle Theorem 4.4.3 with $\tilde{g} = g_{\mathbb{H}}$ to deduce $g_{2,\delta}(t) \ge g_1(t)$ for all $t \in [0, T-\delta]$. Since $g_{2,\delta}(t)$ depends continuously on δ for any $t \in [0, T-\delta]$ we conclude

$$g_1(t) \le \lim_{\delta \searrow 0} g_{2,\delta}(t) = g_2(t).$$

The inverse inequality for $g_1(t)$ and $g_2(t)$ follows by repeating all steps with swapped roles. □

See Section 5.2 for a discussion of more variants and consequences.

4.5 Existence

Whilst the proof of existence on $\mathbb{C}$ is based on Theorem 4.1.2 from the theory of the *logarithmic fast diffusion equation*, the conformally hyperbolic case on the disc relies on Wan-Xiong Shi's existence result for Ricci flows on complete manifolds with bounded curvature (Theorem 2.4.1). But since our initial surface might not be complete nor have bounded curvature, we are going to construct a sequence of complete metrics with bounded curvature on a (compactly contained) exhaustion of the surface which is pointwise decreasing and converging to the original initial metric. This way we can flow each member of the sequence by Shi's theorem and extend it to an immortal solution by virtue of the results of Section 4.3.3. Moreover, the existence of an appropriate lower barrier (Lemma 4.2.1) will allow to find a limit solution which must be instantaneously complete and (by construction) maximally stretched. The pseudolocality-type result of Chen (Theorem 2.6.1) prevents the loss of the initial condition in the limit.

4.5.1 Existence on the conformally hyperbolic disc

The original proof in [GT11] exploits Topping's existence result (Theorem 1.2.2), which then again relies on Shi's existence result (Theorem 2.4.1) and a refined variant of Hamilton's compactness theorem for sequences of Ricci flows [Ham95a]. The variant presented here omits this intermediate step.

Theorem 4.5.1 (Part of [GT11, Theorem 3.1])**.** *Let g_0 be a conformally hyperbolic metric on the disc $\mathbb{D}$. Then there exists a unique, maximally stretched and instantaneously complete Ricci flow $\big(g(t)\big)_{t\in[0,\infty)}$ with $g(0)=g_0$. Moreover, if $\mathrm{K}_{g_0}\le\kappa_0\in[0,\infty)$,*

$$\mathrm{K}_{g(t)}\le\begin{cases}\dfrac{1}{\kappa_0^{-1}-2t} & \text{for all } t\in\Big[0,\frac{1}{2\kappa_0}\Big) \quad \text{if } \kappa_0>0\\ 0 & \text{for all } t\in\big[0,\infty\big) \quad \text{otherwise.}\end{cases} \tag{4.33}$$

The curvature estimate (4.33) was not included in the original article [GT11, Theorem 3.1], but in TOPPING's earlier work [Top10]; moreover, in the proof of this earlier existence result (Theorem 1.2.2) it was indispensable. However, it is not just a simple consequence of a maximum principle, but it additionally relies on the construction in the proof ($\rightarrow$ (4.38)).

PROOF. Choosing a global complex coordinate z ($\rightarrow$ Remark 2.2.2), we write $g_0=\mathrm{e}^{2u_0}|\mathrm{d}z|^2$ for the initial metric and $g_{\mathbb{H}}=\mathrm{e}^{2h}|\mathrm{d}z|^2$ for the complete hyperbolic metric on the disc $\mathbb{D}$. Let $\big(D_j\big)_{j\in\mathbb{N}}\subset\mathbb{D}$ be a suitable exhaustion of the disc $\mathbb{D}$, such that $D_j\Subset D_{j+1}$ for all $j\in\mathbb{N}$, e.g. $D_j=\mathbb{D}_{1-\frac{1}{j+1}}$, and let $H_j=\mathrm{e}^{2h_j}|\mathrm{d}z|^2$ be the complete hyperbolic metric on the smaller disc D_j.

Furthermore, define $M_j:=\inf\Big\{M>1: g_0\big|_{D_j}\le MH_j\Big\}$ and let $g_{0,j}=\mathrm{e}^{2u_{0,j}}|\mathrm{d}z|^2$ be the smoothed-out 'pointwise maximum' of $g_0\big|_{D_j}$ and the rescaled hyperbolic metric $\frac{1}{j^2}H_j$ of curvature $-j^2$ such that

$$\max\Big\{g_0\big|_{D_j},\frac{1}{j^2}H_j\Big\}\le g_{0,j}\le M_jH_j. \tag{4.34}$$

In fact, following [Top10] we can choose a cut-off function $\Psi\in\mathrm{C}^\infty(\mathbb{R})$ with $\Psi(s)=0$ for $s\in(-\infty,-1]$, $\Psi(s)=s$ for $s\in[1,\infty)$ and $\Psi''(s)\ge0$ for all $s\in\mathbb{R}$, and define the interpolated metric by

$$g_{0,j}:=\mathrm{e}^{2\Psi\left(h_j-\log j-u_0|_{D_j}\right)}g_0\big|_{D_j}. \tag{4.35}$$

Hence, $g_{0,j}$ is complete and has bounded curvature since $\Big\{z\in D_j: g_{0,j}(z)>\frac{1}{j^2}H_j(z)\Big\}\Subset D_j$, in particular we have $\kappa_{0,j}:=\sup_{D_j}\mathrm{K}_{g_{0,j}}<\infty$. Note that the $u_{0,j}$ are non-increasing (as the $\frac{1}{j^2}H_j$ are decreasing) and for all $z\in\mathbb{D}$ and j large enough we have

$$\lim_{j\to\infty}u_{0,j}(z)=u_0(z).$$

Therefore, for each $j\in\mathbb{N}$ we can apply SHI's Existence Theorem 2.4.1 to obtain constants $T_j=T_j(\kappa_{0,j})>0$ and $\kappa_j=\kappa_j(T_j,\kappa_{0,j})<\infty$, and a unique complete Ricci flow $\big(g_j(t)\big)_{t\in[0,T_j]}$ on D_j with $g_j(0)=g_{0,j}$, and $\mathrm{K}_{g(t)}\le\kappa_j$ for all $t\in[0,T_j]$. Since by (4.34) $g_j(0)\le M_jH_j$, we may apply Corollary 4.3.4 to show that each $g_j(t)$ can be extended to exist forever. Define $u_j(t)$ such that $\mathrm{e}^{2u_j(t)}|\mathrm{d}z|^2=g_j(t)$.

Using Corollary 2.3.2 to compare $u_{j+1}\big|_{\overline{D_j}}$ to u_j on D_j, observe that for all $(t,z)\in[0,\infty)\times\mathbb{D}$ and $j\in\mathbb{N}$ sufficiently large such that $z\in D_j$, the sequence $\big(u_j(t,z)\big)_{j\in\mathbb{N}}$ is monotone decreasing. Hence, for any $\mathcal{U}\Subset\mathbb{D}$ and $\tau\in(0,\infty)$, u_j is uniformly bounded above (independently of j) on $[0,\tau]\times\mathcal{U}$ for sufficiently large j. On the other hand in order to obtain a uniform lower bound on u_j near $t=0$, we will follow the ideas of [Top10] and appeal to the pseudolocality-type result Theorem 2.6.1 of CHEN. Since $\mathcal{U}$

is relatively compact and g_0 is smooth, we can choose $r_0, v_0 > 0$ sufficiently small such that for all $p \in \mathcal{U}$ (and sufficiently large j), there holds

(i) $\mathcal{B}_{g_0}(p; r_0) \Subset D_j$, in particular $\mathcal{B}_{g_j(t)}(p; r_0) \Subset D_j$ for all $t \in [0, \tau]$;
(ii) $\left|\mathrm{K}_{g_0}\right| \leq r_0^{-2}$ on $\mathcal{B}_{g_0}(p; r_0)$;
(iii) $\mathrm{vol}_{g_0} \mathcal{B}_{g_0}(p; r_0) \geq v_0 r_0^2$.

Therefore we may apply Theorem 2.6.1 to each such flow $g_j(t)$ and obtain a constant $\varepsilon = \varepsilon(v_0, r_0) \in (0, \tau]$ such that for sufficiently large j and $t \in [0, \varepsilon]$

$$\left|\mathrm{K}_{g_j(t)}\right| \leq 2r_0^{-2} \quad \text{on } \mathcal{U}. \tag{4.36}$$

By inspection of the Ricci flow equation (2.5), this gives us a uniform lower bound for u_j on $[0, \varepsilon] \times \mathcal{U}$ for sufficiently large j. For later times $[\varepsilon, \tau]$, we can switch to the lower barrier $\big((2t)g_{\mathbb{H}}\big)_{t \in (0,\infty)}$ provided by Lemma 4.2.1(i), i.e.

$$h\big|_{D_j} + \frac{1}{2}\log(2t) \leq u_j(t) \tag{4.37}$$

for all $t \in (0, \tau]$ independently of j.

Combining these estimates, we find that we have uniform upper and lower bounds for the decreasing sequence u_j on $[0, \tau] \times \mathcal{U}$ (independent of j, for sufficiently large j) and thus we may apply parabolic regularity to get C^k estimates on the functions u_j (uniform in j, for sufficiently large j) on any compact subset of $[0, \infty) \times \mathbb{D}$. Therefore we may define a smooth function $u : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ by

$$u(t, z) := \lim_{j \to \infty} u_j(t, z),$$

and the corresponding metric flow $g(t) := \mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ must be a smooth Ricci flow, defined for all $t \in [0, \infty)$, with $g(0) = g_0$. By (4.37) we also have $(2t)g_{\mathbb{H}} \leq g(t)$ on $\mathbb{D}$ for all $t \in (0, \infty)$, so $\big(g(t)\big)_{t \in [0,\infty)}$ is instantaneously complete.

To see that $\big(g(t)\big)_{t \in [0,\infty)}$ is maximally stretched, let $\tilde{u} : [0, \varepsilon] \times \mathbb{D} \to \mathbb{R}$ be the conformal factor of any other Ricci flow with $\tilde{u}(0, \cdot) \leq u_0$, then the comparison principle (Corollary 2.3.2) tells us that $\tilde{u}|_{D_j}(t, z) \leq u_j(t, z)$ for all $z \in D_j$ and $t \in [0, \varepsilon]$, and therefore (taking $j \to \infty$) $\tilde{u}(t, z) \leq u(t, z)$ for all $z \in \mathbb{D}$ and $t \in [0, \varepsilon]$. Obviously $\big(g(t)\big)_{t \in [0,\infty)}$ is unique amongst maximally stretched solutions ($\to$ Remark 2.2.8).

Finally, in order to show the upper curvature bound if $\sup_{\mathbb{D}} \mathrm{K}_{g_0} < \infty$, define $\kappa_0 := \sup_{\mathbb{D}} \big[\mathrm{K}_{g_0}\big]_+$. Then we have $\kappa_{0,j} \leq \kappa_0$ for all $j \in \mathbb{N}$, because using the cut-off function Ψ from (4.35) and abbreviating $w_j := h_j - \log j - u_0|_{D_j}$, we estimate similarly to [Top10, Eq. (4.3)]

$$\begin{aligned}
\mathrm{K}_{g_{0,j}} &= -\mathrm{e}^{-2\left(\Psi(w_j) + u_0|_{D_j}\right)} \Delta\big(\Psi(w_j) + u_0|_{D_j}\big) \\
&= -\mathrm{e}^{-2\left(\Psi(w_j) + u_0|_{D_j}\right)} \left(\Psi''(w_j)\big|\mathrm{D}w_j\big|^2 + \Psi'(w_j)\Delta\big(h_j - u_0|_{D_j}\big) + \Delta u_0|_{D_j}\right) \\
&\leq -\mathrm{e}^{-2\left(\Psi(w_j) + u_0|_{D_j}\right)} \left(\Psi'(w_j)\Delta h_j + \big(1 - \Psi'(w_j)\big)\Delta u_0|_{D_j}\right) \\
&= \mathrm{e}^{-2\left(\Psi(w_j) + u_0|_{D_j}\right)} \left(\Psi'(w_j)\left(\mathrm{e}^{2h_j}\mathrm{K}_{H_j}\right) + \big(1 - \Psi'(w_j)\big)\left(\mathrm{e}^{2u_0|_{D_j}}\mathrm{K}_{g_0}\big|_{D_j}\right)\right) \\
&= -j^{-2}\mathrm{e}^{-2(\Psi(w_j) - w_j)}\Psi'(w_j) + \mathrm{e}^{-2\Psi(w_j)}\big(1 - \Psi'(w_j)\big)\mathrm{K}_{g_0}\big|_{D_j}
\end{aligned}$$

$$\leq \left[\mathrm{K}_{g_0}\big|_{D_j}\right]_+ \leq \kappa_0 \tag{4.38}$$

using that $\Psi'' \geq 0$, $\Psi' \in [0,1]$ and $\Psi \geq 0$. Therefore, Proposition 2.5.1 gives a uniform (in j) bound for the curvature of each Ricci flow $\big(g_j(t)\big)_{t\in[0,\infty)}$ which survives in the limit, yielding (4.33). □

4.5.2 Existence on the conformally flat plane

Theorem 4.5.2. *Let g_0 be a conformally flat Riemannian metric on the plane $\mathbb{C}$. Then there exists a unique, maximally stretched and instantaneously complete Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ with $g(0) = g_0$ up to a maximal time $T = \frac{1}{4\pi}\operatorname{vol}_{g_0}\mathbb{C} \leq \infty$. Moreover, if $T < \infty$, then we have*

$$\operatorname{vol}_{g(t)}\mathbb{C} = 4\pi(T-t) \qquad \textit{for all } t \in [0,T). \tag{4.39}$$

Proof. (i) Assume $\operatorname{vol}_{g_0}\mathbb{C} < \infty$. Since $(\mathbb{C}, g_0)$ is conformally flat, we may write $g_0 = v_0|\mathrm{d}z|^2$ for some positive function $0 < v_0 \in \mathrm{C}^\infty(\mathbb{C})$. Note that $\left\|v_0\right\|_{\mathrm{L}^1(\mathbb{C})} = \operatorname{vol}_{g_0}\mathbb{C}$. The existence Theorem 4.1.2 for the logarithmic fast diffusion equation provides a solution $v \in \mathrm{C}^\infty\big((0,T)\times\mathbb{C}\big) \cap \mathrm{C}\big([0,T),\mathrm{L}^1(\mathbb{C})\big)$ with $v(0) = v_0$; in particular, $\big(g(t)\big)_{t\in[0,T)} = \big(v(t)|\mathrm{d}z|^2\big)_{t\in[0,T)}$ is a Ricci flow on $\mathbb{C}$ with $g(0) = g_0$. Integrating the square root of $v(t,\cdot)$ along any rays and using its minimal decay (4.7), we see that $\big(g(t)\big)_{t\in[0,T)}$ is instantaneously complete. Hence, by Corollary 4.4.2 it is also maximally stretched. Finally it satisfies the volume equation (4.39) which corresponds to (4.6).*

(ii) In the case of infinite volume $\operatorname{vol}_{g_0}\mathbb{C} = \infty$ we are going to approximate the solution by a sequence of finite volume solutions: Define a weakly increasing sequence $\big(g_{0,j}\big)_{j\in\mathbb{N}}$ that converges smoothly locally to g_0, but has finite volume $\operatorname{vol}_{g_{0,j}}\mathbb{C} < \infty$ for each $j \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, by the first part (i) we have an instantaneously complete and maximally stretched Ricci flow $g_j(t)$ with $g_j(0) = g_{0,j}$, defined for all $t \in [0, T_j)$ up to a maximal time $T_j = \frac{1}{4\pi}\operatorname{vol}_{g_{0,j}}\mathbb{C} \to \infty$ as $j \to \infty$. Since each solution is maximally stretched and $\big(g_j(0)\big)_{j\in\mathbb{N}}$ is weakly increasing, the sequence $\big(g_j(t)\big)_{j\in\mathbb{N}}$ is also weakly increasing for all $j \geq j_0$ such that $t \leq T_{j_0}$. By Lemma 4.2.1(ii) there is also a uniform (in j) upper barrier on compact subsets of $[0,\infty)\times\mathbb{C}$, hence by parabolic regularity theory, the sequence $\big(g_j(t)\big)_{j\in\mathbb{N}}$ converges locally smoothly on $[0,\infty)\times\mathbb{C}$ to a smooth Ricci flow $\big(g(t)\big)_{t\in[0,\infty)}$ with $g(0) = g_0$. Since for all $t \in (0,\infty)$ we have $g(t) \geq g_j(t)$ for all j sufficiently large so that $T_j > t$, we find that $g(t)$ is also complete. Hence, $\big(g(t)\big)_{t\in[0,\infty)}$ is instantaneously complete and hence by Corollary 4.4.2 also maximally stretched. □

Note that the construction in the proof of Theorem 4.5.1 for the conformally hyperbolic solution could also provide the existence of a Ricci flow on $\mathbb{C}$ if one found a replacement for the required lower barrier (Lemma 4.2.1(i)). However, the solution $\big(g(t)\big)_{t\in[0,T)}$ by Theorem 4.5.2 is such a barrier, and therefore we obtain the curvature bound (4.33) as in the conformally hyperbolic case (or even in the original Theorem 1.2.2 by Topping).

Corollary 4.5.3. *For maximal $T > 0$ let $\big(g(t)\big)_{t\in[0,T)}$ be a conformally flat and instantaneously complete Ricci flow on the plane $\mathbb{C}$. If the Gaussian curvature is initially*

*Alternatively we could use Huber's generalisation (Theorem A.1.2) of the Gauss-Bonnet theorem to non-compact but complete surfaces with finite area to see that $\int_{\mathbb{C}} \mathrm{K}_{g(t)}\mathrm{d}\mu_{g(t)} = 2\pi$ for all $t \in (0,T)$. Like in the compact case (→ Lemma 3.3.1), the volume equation (4.39) follows from (2.15).

bounded from above, say $\mathrm{K}_{g(0)} \leq \kappa_0$ *for some* $\kappa_0 < \infty$*, then we have the curvature estimate* (4.33)*.*

4.6 Long-time behaviour on the disc

Finally, we would like to analyse the long-time behaviour of instantaneously complete Ricci flows on non-compact surfaces. Whilst the *a priori* estimates from Section 4.3.2 suggest convergence of conformally hyperbolic solutions to the hyperbolic metric, the situation in the conformally Euclidean case on the plane (or on the infinite cylinder) turns out to be rather difficult. For instance, one should only compare the behaviour of a finite area solution which ends in finite time with the lifted flow from a torus, where the Ricci flow does uniformise ($\rightarrow$ Theorem 3.1.1). For a more detailed discussion and a compilation of known results with references, we refer to Section 5.3.2 in the next chapter.

We conclude this chapter with the following result which characterises the long-time behaviour of conformally hyperbolic instantaneously complete Ricci flows on the disc. Note that, we do not require the solution to be maximally stretched.

Theorem 4.6.1 (Part of [GT11, Theorem 3.1]). *If* $\big(g(t)\big)_{t\in[0,\infty)}$ *is an instantaneously complete and conformally hyperbolic Ricci flow on the complete hyperbolic disc* $(\mathbb{D}, g_{\mathbb{H}})$*, then the rescaled flow converges smoothly locally*

$$\frac{1}{2t}g(t) \longrightarrow g_{\mathbb{H}} \quad \text{as} \quad t \to \infty. \tag{4.40}$$

Proof. For any $r \in (0,1]$, let H_r be the complete hyperbolic metric on the disc $\mathbb{D}_r$ of radius r and $M_r := \inf\big\{M > 0 : g(0)\big|_{\mathbb{D}_r} \leq MH_r\big\}$. For fixed $\mathcal{U} \Subset \mathbb{D}$ define $\delta := \operatorname{dist}(\mathcal{U}, \partial\mathbb{D}) > 0$, and fix $k \in \mathbb{N}$. Using Lemma B.3.2 (with constant $C' = C'(\delta, k) > 0$) and Lemma 4.3.1 with $g(t) = \mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ we establish uniform C^k-bounds on $\mathcal{U}$ for all $t \geq 1$

$$\begin{aligned}\sup_{\mathcal{U}} \left|{}^{g_{\mathbb{H}}}\nabla^k\left(\frac{1}{2t}g(t) - g_{\mathbb{H}}\right)\right|_{g_{\mathbb{H}}} &= \sup_{\mathcal{U}} \left|{}^{g_{\mathbb{H}}}\nabla^k\left(\frac{1}{2t}g(t)\right)\right|_{g_{\mathbb{H}}} \leq C' \left\|\frac{1}{2t}g(t)\right\|_{\mathrm{C}^k(\mathbb{D}_{1-\delta})} \\ &= \sqrt{2}C' \left\|\mathrm{e}^{2\left(u(t) - \frac{1}{2}\log 2t\right)}\right\|_{\mathrm{C}^k(\mathbb{D}_{1-\delta})} \\ &\leq C\Big(k, \delta, M_{1-\delta/2}\Big).\end{aligned} \tag{4.41}$$

Note that for $0 < s \leq r \leq 1$ we have $g_{\mathbb{H}}\big|_{\mathbb{D}_s} \leq H_r\big|_{\mathbb{D}_s} \leq H_s$. Using the barriers from Lemma 4.2.1 we can estimate for any $r \in (1-\delta, 1)$ on $\mathbb{D}_r \Supset \mathcal{U}$

$$0 \leq \frac{1}{2t}g(t) - g_{\mathbb{H}} = \left(\frac{1}{2t}g(t) - H_r\right) + \big(H_r - g_{\mathbb{H}}\big) \leq \frac{M_r}{2t}H_r + \big(H_r - g_{\mathbb{H}}\big).$$

Therefore

$$\begin{aligned}\limsup_{t\to\infty} \left\|\frac{1}{2t}g(t) - g_{\mathbb{H}}\right\|_{\mathrm{C}^0(\mathcal{U}, g_{\mathbb{H}})} &\leq \limsup_{t\to\infty} \frac{M_r}{2t}\Big\|H_r\Big\|_{\mathrm{C}^0(\mathcal{U}, g_{\mathbb{H}})} + \Big\|H_r - g_{\mathbb{H}}\Big\|_{\mathrm{C}^0(\mathcal{U}, g_{\mathbb{H}})} \\ &= \Big\|H_r - g_{\mathbb{H}}\Big\|_{\mathrm{C}^0(\mathcal{U}, g_{\mathbb{H}})} \quad \xrightarrow{r \nearrow 1} \quad 0.\end{aligned} \tag{4.42}$$

Combining the local uniform C^k-bounds (4.41) with the C^0-convergence (4.42) we obtain local convergence in the C^k-norm

$$\left\| \frac{1}{2t} g(t) - g_{\mathbb{H}} \right\|_{\mathrm{C}^k(\mathcal{U}, g_{\mathbb{H}})} \quad \overset{t\to\infty}{\longrightarrow} \quad 0. \qquad \square$$

Chapter 5

Ricci flow on arbitrary surfaces

In this chapter we are going to generalise the results of existence, uniqueness and long-time behaviour from simply connected surfaces in the previous two chapters to surfaces with arbitrary topology. The key observation is that almost all the results for a lifted solution on the (simply connected) universal cover of a given surface directly transfer to the Ricci flow on the base; merely the issue of existence requires a little more caution.

5.1 Existence

A direct consequence of the uniqueness property of maximally stretched Ricci flows (Remark 2.2.8) is the preservation of isometries under the flow.

Lemma 5.1.1. *Let $\big(g(t)\big)_{t\in[0,T]}$ be a maximally stretched Ricci flow on a surface $\mathcal{M}^2$. Then the isometry group does not shrink under the flow:* $\operatorname{Isom}\big(\mathcal{M}, g(0)\big) \subset \operatorname{Isom}\big(\mathcal{M}, g(t)\big)$ *for all $t \in [0,T]$.*

Proof. Pick any $\psi \in \operatorname{Isom}\big(\mathcal{M}, g(0)\big)$. Since the Ricci flow is invariant under diffeomorphisms, $\psi^* g(t)$ is again a solution to the Ricci flow with $\psi^* g(0) = g(0)$ for all $t \in [0,T]$. Because $g(t)$ is maximally stretched, we have $\psi^* g(t) \leq g(t)$ and also $(\psi^{-1})^* g(t) \leq g(t)$ for all $t \in [0,T]$. Pulling back the latter inequality by ψ yields $g(t) \leq \psi^* g(t)$. Therefore $g(t) = \psi^* g(t)$ and $\psi \in \operatorname{Isom}\big(\mathcal{M}, g(t)\big)$ for all $t \in [0,T]$. $\square$

Using the universal cover, we may now reduce the proof of existence of instantaneously complete Ricci flows on arbitrary surfaces to simply connected ones, which we have dealt with throughly in the previous chapters.

Theorem 5.1.2 (Part of [GT11, Theorem 1.3])**.** *Let $\big(\mathcal{M}^2, g_0\big)$ be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal class, we define $T \in (0,\infty]$ by*

$$T := \begin{cases} \dfrac{\operatorname{vol}_{g_0} \mathcal{M}}{4\pi\,\chi(\mathcal{M})} & \text{if } \big(\mathcal{M}, g_0\big) \text{ is not conformally hyperbolic and } \chi(\mathcal{M}) > 0^*, \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

Then there exists a smooth Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ such that

*This includes the sole cases that $\big(\mathcal{M}, g_0\big)$ is conformally equivalent either to the round sphere $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$, the projective plane $\big(\mathbb{R}P^2, g_{\mathbb{S}}\big)$ or the Euclidean plane $\big(\mathbb{C}, g_{\mathbb{E}}\big)$. Note that in the latter case, also $T = \infty$ if $\operatorname{vol}_{g_0} \mathbb{C} = \infty$.

(i) $g(0) = g_0$;

(ii) $\big(g(t)\big)_{t\in[0,T)}$ *is instantaneously complete;*

(iii) $\big(g(t)\big)_{t\in[0,T)}$ *is maximally stretched,*

and this flow is unique in the sense that if $\big(g_2(t)\big)_{t\in[0,T_2)}$ *is any other Ricci flow on* $\mathcal{M}$ *satisfying (i) and (iii), then* $T_2 \le T$ *and* $g_2(t) = g(t)$ *for all* $t \in [0, T_2)$.

If $T < \infty$*, then we have*

$$\operatorname{vol}_{g(t)} \mathcal{M} = 4\pi\,\chi(\mathcal{M})\,(T-t) \quad \longrightarrow \quad 0 \qquad \text{as } t \nearrow T, \tag{5.2}$$

and in particular, T *is the maximal existence time. Moreover, if* $\mathrm{K}_{g_0} \le \kappa_0 < \infty$*, then*

$$\mathrm{K}_{g(t)} \le \begin{cases} \dfrac{1}{1/\kappa_0 - 2t} & \text{for all } t \in \big[0, \tfrac{1}{2\kappa_0}\big) \quad \text{if } \kappa_0 > 0, \\ 0 & \text{for all } t \in \big[0, \infty\big) \quad \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a universal covering of $\mathcal{M}$ with the lifted metric $\widetilde{g}_0 = \pi^* g_0$, and let $\Gamma := \operatorname{Deck}(\pi) < \operatorname{Isom}(\widetilde{\mathcal{M}}, \widetilde{g}_0)$ be the associated discrete subgroup of the isometry group, isomorphic to the fundamental group $\pi_1(\mathcal{M})$, such that $\mathcal{M} \cong \widetilde{\mathcal{M}}/\Gamma$. Then by virtue of the Uniformisation Theorem A.2.4, $\big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big)$ is conformally equivalent either to the round sphere $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$, to the flat plane $\big(\mathbb{C}, g_{\mathbb{E}}\big)$ or to the hyperbolic disc $\big(\mathbb{D}, g_{\mathbb{H}}\big)$.

Therefore by Theorems 3.3.2, 4.5.2 and 4.5.1 respectively there exists an instantaneously complete and maximally stretched Ricci flow $\big(\widetilde{g}(t)\big)_{t\in[0,T)}$ on $\widetilde{\mathcal{M}}$ with $\widetilde{g}(0) = \widetilde{g}_0$ up to a maximal time

$$T = \begin{cases} \frac{1}{8\pi} \operatorname{vol}_{\widetilde{g}_0} \widetilde{\mathcal{M}} & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{S}^2, g_{\mathbb{S}}\big), \\ \frac{1}{4\pi} \operatorname{vol}_{\widetilde{g}_0} \widetilde{\mathcal{M}} & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{C}, g_{\mathbb{E}}\big), \\ \infty & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{D}, g_{\mathbb{H}}\big). \end{cases}$$

By Lemma 5.1.1, Γ acts by isometries on $\big(\widetilde{\mathcal{M}}, \widetilde{g}(t)\big)$ for every $t \in [0, T)$, so we may quotient $\widetilde{g}(t)$ to obtain uniquely a maximally stretched and instantaneously complete solution $g(t) = \pi_* \widetilde{g}(t)$ on $\mathcal{M} \cong \widetilde{\mathcal{M}}/\Gamma$ for all $t \in [0, T)$ with $g(0) = g_0$. Using the relation $|\Gamma| \cdot \operatorname{vol}_{g_0} \mathcal{M} = \operatorname{vol}_{\widetilde{g}_0} \widetilde{\mathcal{M}}$ we are going to phrase the maximal existence time T in terms of $\big(\mathcal{M}, g_0\big)$ by distinguishing the only cases:

$$T = \begin{cases} \frac{1}{8\pi} \operatorname{vol}_{g_0} \mathcal{M} & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{S}^2, g_{\mathbb{S}}\big) \text{ and } |\Gamma| = 1 & \Longrightarrow & \big(\mathcal{M}, g_0\big) \cong \big(\mathbb{S}^2, g_{\mathbb{S}}\big) \\ \frac{1}{4\pi} \operatorname{vol}_{g_0} \mathcal{M} & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{S}^2, g_{\mathbb{S}}\big) \text{ and } |\Gamma| = 2 & \Longrightarrow & \big(\mathcal{M}, g_0\big) \cong \big(\mathbb{R}P^2, g_{\mathbb{S}}\big) \\ \frac{1}{4\pi} \operatorname{vol}_{g_0} \mathcal{M} & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{C}, g_{\mathbb{E}}\big) \text{ and } |\Gamma| = 1 & \Longrightarrow & \big(\mathcal{M}, g_0\big) \cong \big(\mathbb{C}, g_{\mathbb{E}}\big) \\ \infty & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{C}, g_{\mathbb{E}}\big) \text{ and } |\Gamma| = \infty \\ \infty & \text{if } \big(\widetilde{\mathcal{M}}, \widetilde{g}_0\big) \cong \big(\mathbb{D}, g_{\mathbb{H}}\big). \end{cases}$$

If $T < \infty$, then $|\Gamma| < \infty$, and the volume equation (5.2) descends from the corresponding ones (3.20) and (4.39) for the lifted Ricci flow divided by $|\Gamma| = \frac{\chi(\widetilde{\mathcal{M}})}{\chi(\mathcal{M})}$:

$$\operatorname{vol}_{g(t)} \mathcal{M} = \frac{1}{|\Gamma|} \operatorname{vol}_{\widetilde{g}(t)} \widetilde{\mathcal{M}} = \frac{4\pi\,\chi(\widetilde{\mathcal{M}})}{|\Gamma|}\,(T-t) = 4\pi\,\chi(\mathcal{M})(T-t) \qquad \text{for all } t \in [0, T).$$

Finally, if $K_{g_0} \le \kappa_0 < \infty$ the curvature estimate (5.3) follows from Proposition 2.5.1 if

$\mathcal{M}$ is compact, and from Theorem 4.5.1 and Corollary 4.5.3 otherwise. $\square$

5.2 Uniqueness

The uniqueness of the solution in Theorem 5.1.2 originates from the requirement (iii) that it is maximally stretched. A more interesting question would be, whether the solution is also *unique* within the class of instantaneously complete Ricci flows starting at g_0. Using the characterisation of solutions of Theorem 5.1.2 by the property that they are maximally stretched, we can rephrase the issue of uniqueness conjectured by Topping from [Top10] as follows.

Conjecture 5.2.1 (Topping [Top10])**.** Every instantaneously complete Ricci flow on a surface is maximally stretched.

Whilst the conjecture is trivial for Ricci flows on compact surfaces by virtue of Hamilton's uniqueness Theorem 2.4.1, the other non-hyperbolic cases were proved first in [GT11] where the authors exploited a comparison principle of Rodriguez, Vazquez and Esteban for the *logarithmic fast diffusion equation* [RVE97] as we have described in detail in Section 4.4.1.

Theorem 5.2.2 ([GT11, Theorem 1.6])**.** *Suppose $\big(g(t)\big)_{t\in[0,T]}$ is an instantaneously complete Ricci flow on the Riemannian surface $\big(\mathcal{M}, g(0)\big)$ which does* not *admit any complete hyperbolic metric. Then $\big(g(t)\big)_{t\in[0,T]}$ is maximally stretched.*

Proof. Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a universal covering. Then, by virtue of the uniformisation theorem (Corollary A.2.5), $\big(\widetilde{\mathcal{M}}, \pi^* g(0)\big)$ is conformally equivalent either to the round sphere $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$ or to the complex plane $\big(\mathbb{C}, g_{\mathbb{E}}\big)$. Therefore, the lifted, still instantaneously complete Ricci flow $\big(\pi^* g(t)\big)_{t\in[0,T]}$ is maximally stretched by virtue of Corollaries 3.3.3 and 4.4.2 respectively. But then the original Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on the base $\mathcal{M}$ has to be maximally stretched, too. $\square$

Note that no curvature assumption is made on $g(t)$ in the theorem above. This can be compared to a result of Chen which would imply this type of *strong uniqueness* in the case that $(\mathcal{M}, g_0)$ is complete, of bounded curvature and with controlled geometry (see Theorem 5.2.7 below).

5.2.1 Conformally hyperbolic Ricci flows

In order to show that an instantaneously complete and conformally hyperbolic Ricci flow corresponds to the solution provided by Theorem 5.1.2, up to now we have to make two additional assumptions: Initially it has to be bounded from above by a multiple of the hyperbolic metric and the curvature has to be controlled from above for a short time.

Theorem 5.2.3. *Let $\big(g(t)\big)_{t\in[0,T]}$ be an instantaneously complete Ricci flow on a hyperbolic surface $\big(\mathcal{M}^2, g_{\mathbb{H}}\big)$ such that $g(0)$ is conformally equivalent to $g_{\mathbb{H}}$. Suppose that for some constants $M > 0$ and $\varepsilon \in (0, T]$ and some integrable $f \in \mathrm{L}^1\big([0,\varepsilon)\big)$, we have $g(0) \le M g_{\mathbb{H}}$ and $\mathrm{K}_{g(t)} \le f(t)$ for all $t \in [0,\varepsilon)$, then $\big(g(t)\big)_{t\in[0,T]}$ is maximally stretched.*

PROOF. Let $\big(G(t)\big)_{t\in[0,\infty)}$ be the maximally stretched Ricci flow with $G(0) = g(0)$ provided by Theorem 5.1.2. Therefore by (5.3), there is some $\tau \in (0, T]$ such that $\sup_{\mathcal{M}} \mathrm{K}_{G(t)} < \infty$ for all $t \in [0, \tau]$ because $\mathrm{K}_{G(0)} \le f(0)$. Hence we can apply Theorem 4.4.4 to both flows and conclude the theorem's statement. □

The following is a slight generalisation of a result from [GT10], where we require the initial metric to have strictly negative curvature, which implies the upper barrier by a multiple of the hyperbolic metric.

Corollary 5.2.4. *For some constants $T > 0$, $\eta > 0$ and $\varepsilon \in (0, T]$ and some integrable $f \in \mathrm{L}^1\big((0,\varepsilon)\big)$, let $\big(g(t)\big)_{t\in[0,T]}$ be an instantaneously complete Ricci flow on a surface $\mathcal{M}^2$ such that $\mathrm{K}_{g(0)} \le -\eta$ and $\mathrm{K}_{g(t)} \le f(t)$ for all $t \in (0, \varepsilon)$. Then $\big(g(t)\big)_{t\in[0,T]}$ is maximally stretched.*

5.2.2 Relation to the *classical* Ricci flow

The contribution of Chen and Zhu [CZ06] to Theorem 2.4.1 was the uniqueness in the complete case with bounded curvature. In our situation where the underlying manifold is two-dimensional, this is much simpler to prove. The statement and proof are as follows.

Theorem 5.2.5. *Let $\big(g_1(t)\big)_{t\in[0,T]}$ and $\big(g_2(t)\big)_{t\in[0,T]}$ be two complete Ricci flows on a surface $\mathcal{M}^2$, with uniformly bounded curvature $\big|\mathrm{K}_{g_i(t)}\big| \le \kappa < \infty$ for all $t \in [0, T]$ and $i \in \{1, 2\}$. If $g_1(0) = g_2(0)$, then $g_1(t) = g_2(t)$ for all $t \in [0, T]$.*

PROOF. Since $g_1(t)$ and $g_2(t)$ are conformally equivalent, we may define $Q \in \mathrm{C}^\infty\big([0,T]\times \mathcal{M}\big)$ via $g_1(t) = \mathrm{e}^{2Q(t)} g_2(t)$ for all $t \in [0, T]$. Differentiating this relation ($\to$ (4.31)) yields $\frac{\partial}{\partial t} Q(t) = \mathrm{K}_{g_2(t)} - \mathrm{K}_{g_1(t)}$. Since $Q(0) = 0$ and the curvatures are uniformly bounded by κ, we have

$$\left|\frac{\partial}{\partial t} Q(t)\right| = \left|\mathrm{K}_{g_2(t)} - \mathrm{K}_{g_1(t)}\right| \le 2\kappa \quad \Longrightarrow \quad \big|Q(t)\big| \le 2\kappa t$$

for all $t \in [0, T]$. Because both g_1 and g_2 are complete and have uniformly bounded curvature, we may apply twice Theorem 4.4.3 taking either $\tilde{g} = \min_{t\in[0,T]} g_1(t)$ or $\tilde{g} = \min_{t\in[0,T]} g_2(t)$ to obtain $g_2(t) \le g_1(t)$ and $g_1(t) \le g_2(t)$ respectively for all $t \in [0, T]$. □

Since the uniqueness Conjecture 5.2.1 has not been fully resolved, it is *a priori* conceivable that in the case that $(\mathcal{M}, g_0)$ is complete and of bounded curvature, the Ricci flow we construct in Theorem 5.1.2 could be different from the classical solution of Theorem 2.4.1 by Hamilton or Shi. In fact, one could think of a complete, conformally hyperbolic Ricci flow with uniformly bounded curvature starting from a metric which is not bounded from above by a multiple of the hyperbolic metric, e.g. $\frac{1}{(1-|z|^2)^4}|\mathrm{d}z|^2$ on the disc $\mathbb{D}$, which is certainly not covered by Theorem 5.2.3. However, we rule out this possibility in the following corollary.

Corollary 5.2.6 ([GT11, Theorem 1.8])**.** *Every complete Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on a surface $\mathcal{M}^2$ with bounded curvature is maximally stretched.*

PROOF. Let $\kappa := \max_{t\in[0,T]} \sup_{\mathcal{M}} \big|\mathrm{K}_{g(t)}\big| < \infty$ be the uniform bound for the curvature of $\big(g(t)\big)_{t\in[0,T]}$, and let $\big(G(t)\big)_{t\in[0,T]}$ be the complete and maximally stretched Ricci flow from Theorem 5.1.2 starting at $G(0) = g(0)$. Note that by CHEN's Theorem 2.5.2 we have the uniform lower bound to the curvature $K_{G(t)} \geq -\kappa$ for all $t \in [0, T]$. If $\kappa \leq 0$ or $T < \frac{1}{2\kappa}$, then estimate (5.3) already tells us that $\sup_{\mathcal{M}} \mathrm{K}_{G(t)} < \infty$ for all $t \in [0, T]$, and the statement follows from Theorem 5.2.5.

Otherwise, we can inductively apply Theorems 5.2.5 and 5.1.2 to extend the upper curvature bound of $G(t)$ to the whole time interval $[0, T]$: Assume for some $\tau \in [0, T)$ we have $\sup_{\mathcal{M}} \mathrm{K}_{G(t)} \leq \kappa$ for all $t \in [0, \tau]$, then we can apply Theorem 5.1.2 to $G(\tau)$ and obtain a new Ricci flow which coincides with $G(t)$ but from (5.3) it has a new curvature bound $\sup_{\mathcal{M}} \mathrm{K}_{G(t)} < \infty$ for all $t \in \big[0, \min\{T, \tau + 1/2\kappa\}\big)$. By Theorem 5.2.5, we have $g(t) = G(t)$ and in particular $\mathrm{K}_{G(t)} = \mathrm{K}_{g(t)} \leq \kappa$ for all $t \in \big[0, \min\{T, \tau + 1/2\kappa\}\big]$. Gradually repeating this step, we obtain $g(t) = G(t)$ for all $t \in [0, T]$. □

Finally, Corollary 5.2.6 allows us to state CHEN's *strong uniqueness* result from [Che09] in terms of Conjecture 5.2.1. It is a direct consequence of the curvature bounds (Corollary 2.6.2) obtained by a more elaborated variant of PERELMAN's pseudolocality.

Theorem 5.2.7 (CHEN [Che09, Theorem 3.10])**.** *For some constants $\kappa_0 < \infty$ and $v_0 > 0$ let $\big(g(t)\big)_{t\in[0,T]}$ be a complete Ricci flow on a surface $\mathcal{M}^2$ such that $\big|\mathrm{K}_{g(0)}\big| \leq \kappa_0$ and $\mathrm{vol}_{g(0)} \mathcal{B}_{g(0)}(p; 1) \geq v_0$ for all $p \in \mathcal{M}$. Then $\big(g(t)\big)_{t\in\left[0,\min\{T,\kappa_0^{-1}\}\right]}$ is maximally stretched.*

5.3 Long-time behaviour

In Section 3.1 we have summarised HAMILTON and CHOW's classical result about the asymptotic behaviour of the normalised Ricci flow on *compact* surfaces. Together with the results in the conformally hyperbolic case on *non-compact* surfaces (→ §4.3.2 and §4.6), we can summarise:

Theorem 5.3.1 ([Ham88, Theorem 1.3], [Cho91, Theorem 1.2]; [GT11, Theorem 1.3])**.** *Suppose $\big(g(t)\big)_{t\in[0,T)}$ is an instantaneously complete Ricci flow with maximal existence time $T \in (0, \infty]$ according to (5.1) on a surface $\mathcal{M}^2$ such that $\big(\mathcal{M}^2, g(0)\big)$ is* not *conformally Euclidean*[†] *then the rescaled flow*

$$\begin{cases} \dfrac{1}{2(T-t)}\, g(t) & \text{if } g(0) \text{ is conformally spherical,} \\[2ex] \dfrac{1}{2t}\, g(t) & \text{if } g(0) \text{ is conformally hyperbolic} \end{cases}$$

converges smoothly and locally to the complete conformal metric $g_{\mathbb{S}}$ or $g_{\mathbb{H}}$ of constant curvature 1 or -1 respectively as $t \nearrow T$. Moreover, if $g(0)$ is either conformally spherical or bounded from above by a multiple of the hyperbolic metric, i.e. for some $M > 0$ we have $g(0) \leq M g_{\mathbb{H}}$, then the convergence is global: For any $k \in \mathbb{N}_0$ and

[†] Recall that we denote a surface to be conformally Euclidean if it is conformally equivalent to the flat plane $(\mathbb{C}, g_{\mathbb{E}})$ or to the flat cylinder $(\mathbb{R} \times \mathbb{S}^1, g_{\mathbb{E}})$.

$\eta \in (0,1)$ there is a constant $C = C(k,\eta,g(0)) > 0$[‡] *such that*

$$\begin{aligned} &\left\| \frac{1}{2(T-t)} g(t) - g_{\mathbb{S}} \right\|_{\mathrm{C}^k(\mathcal{M},g_{\mathbb{S}})} \le C(T-t)^{1-\eta} && \textit{for all } t \in [0,T), \\ \textit{or} \quad &\left\| \frac{1}{2t} g(t) - g_{\mathbb{H}} \right\|_{\mathrm{C}^k(\mathcal{M},g_{\mathbb{H}})} \le \frac{C}{t^{1-\eta}} && \textit{for all } t \in [1,\infty) \end{aligned} \tag{5.4}$$

respectively.

Proof. Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be the universal covering with $\widetilde{\mathcal{M}} \in \{\mathbb{S}^2, \mathbb{D}\}$ and let $\big(\widetilde{g}(t)\big)_{t\in[0,T)} = \big(\pi^* g(t)\big)_{t\in[0,T)}$ be the lifted Ricci flow. If $\widetilde{\mathcal{M}} = \mathbb{S}^2$, we already have global (and local) smooth convergence to the spherical space form on $\mathbb{S}^2$ and therefore also for the unlifted flow on the base $\mathcal{M} \in \{\mathbb{S}^2, \mathbb{R}P^2\}$ by virtue of Corollary 3.4.6.

On the other hand if $\widetilde{\mathcal{M}} = \mathbb{D}$, in order to show local convergence on an arbitrary ball $\mathcal{B}_{g_{\mathbb{H}}}(p;r)$ in $\big(\mathcal{M}, g_{\mathbb{H}}\big)$, choose a point $\widetilde{p} \in \pi^{-1}(\{p\})$ and consider the ball $\mathcal{B}_{\pi^* g_{\mathbb{H}}}(\widetilde{p};r) \subset \mathbb{D}$. Then the local smooth convergence of $\frac{1}{2t}g(t)$ to $g_{\mathbb{H}}$ on $\mathcal{B}_{g_{\mathbb{H}}}(p;r)$ is a consequence of Theorem 4.6.1 which we can apply to show smooth convergence of $\frac{1}{2t}\widetilde{g}(t)$ to $\pi^* g_{\mathbb{H}}$ on $\mathcal{B}_{\pi^* g_{\mathbb{H}}}(\widetilde{p};r)$. The global convergence in the case that $g(0) \le M g_{\mathbb{H}}$ for some $M > 0$ is the statement of Theorem 4.3.2. □

Using Definiton 2.2.4 of the normalised Ricci flow and its conversions (3.4) and (4.15), we rewrite the above statement in the following unifying way:

Corollary 5.3.2. *Suppose $\big(\bar{g}(\bar{t})\big)_{\bar{t}\in[0,\infty)}$ is a complete normalised Ricci flow on a surface $\mathcal{M}^2$ such that $\big(\mathcal{M}^2, \bar{g}(0)\big)$ is* not *conformally Euclidean, then it converges smoothly and locally to the complete conformal metric $g_{\bar{\kappa}} \in \{g_{\mathbb{S}}, g_{\mathbb{H}}\}$ of constant curvature $\mathrm{K}_{g_{\bar{\kappa}}} \equiv \bar{\kappa} \in \{1,-1\}$ respectively as $\bar{t} \nearrow \infty$. Moreover, if $\bar{g}(0)$ is either conformally spherical or bounded from above by a multiple of the hyperbolic metric, i.e. for some $M > 0$ we have $\bar{g}(0) \le M g_{\mathbb{H}}$, then the convergence is global: For any $k \in \mathbb{N}_0$ and $\eta \in (0,1)$ there is a constant $C = C(k,\eta,\bar{g}(0)) > 0$ such that*

$$\left\| \bar{g}(\bar{t}) - g_{\bar{\kappa}} \right\|_{\mathrm{C}^k(\mathcal{M},g_{\bar{\kappa}})} \le C\mathrm{e}^{-2(1-\eta)\bar{t}} \qquad \textit{for all } t \in [1,\infty). \tag{5.5}$$

Remark 5.3.3. For the analysis of the long-time behaviour it is not important whether the initial metric is complete or not. Moreover, by the conversion of the conformally hyperbolic unnormalised flow via (4.15), the associated normalised solution will be eternal. Hence, at time $\bar{t} = 0$ it is complete and will be still below some multiple of the hyperbolic metric, provided that this is true for initial unnormalised metric.

5.3.1 Further results about conformally hyperbolic Ricci flows

The combination of Theorem 5.1.2, Corollary 5.2.6 and Theorem 5.3.1 generalises a number of other results which have appeared recently, proved with different techniques:

- In the special case that the initial surface $\big(\mathcal{M}, g_0\big)$ is complete, topologically finite, with negative Euler characteristic $\chi(\mathcal{M}) < 0$, and on each end of $\mathcal{M}$ the initial metric g_0 is asymptotic to a multiple of a hyperbolic cusp metric [JMS09] or of a funnel metric [AAR09], Lizhen Ji, Rafe Mazzeo and Natasa Šešum or Pierre Albin, Clara L. Aldana and Frédéric Rochon respectively show

[‡]In fact, in the hyperbolic case C depends on the constant M rather than the whole metric $g(0)$.

existence and exponentially fast, smooth uniform convergence of the normalised Ricci flow to the unique complete metric of constant curvature in the conformal class of g_0. Since the number of cusp or funnel ends is finite and their complement is compact we observe that there exists a conformally equivalent metric $Mg_{\mathbb{H}}$ of constant negative curvature with $g_0 \leq Mg_{\mathbb{H}}$ and we may apply alternatively Theorem 5.1.2 and Corollary 5.3.2.

- If $\left(\mathcal{M}^2, g_0\right)$ is a complete Riemannian surface with asymptotically conical ends and negative Euler characteristic $\chi(\mathcal{M}) < 0$, then James Isenberg, Mazzeo and Šešum show in [IMS12] the existence of a Ricci flow $\left(g(t)\right)_{t\in[0,\infty)}$ on $\mathcal{M}$, with $g(0) = g_0$, and smooth local convergence of the rescaled flow $t^{-1}g(t)$ to a complete metric of constant negative curvature and finite area in the conformal class of g_0. This is a special case of Theorems 5.1.2 and 5.3.1.
- For an initial metric g_0 on the disc $\mathbb{D}$ which is bounded above and below by positive multiples of the complete hyperbolic metric $g_{\mathbb{H}}$, Oliver C. Schnürer, Felix Schulze and Miles Simon show existence and exponentially fast, smooth uniform convergence of the normalised Ricci flow to $g_{\mathbb{H}}$ in [SSS11]. Theorem 5.1.2 and Corollary 5.3.2 imply that the metrical equivalence[§] can be weakened to $g_0 \leq Mg_{\mathbb{H}}$ for some $M > 0$. Moreover, without this condition we still have smooth local convergence.

5.3.2 Conformally Euclidean solutions

The situation in the conformally Euclidean case is much more involved. First, in contrast to the non-flat cases, where one can choose the curvature of the limiting space form up to a sign, there is no natural scaling for a flat metric on $\mathbb{C}$. Whereas the Ricci flow uniformises *compact* conformally Euclidean surfaces towards the flat metric of the same volume, on the non-compact plane (or even on the infinite cylinder) several different phenomena arise: If the volume is finite the flow ends in finite time (Theorem 5.1.2); on the other hand we have the stationary flat solution $g_{\mathbb{E}}$ and Hamilton's cigar soliton g_Σ (→ § 6.2.1). Whilst both exist for all time, the Ricci flow of the latter one is self-similar and hence not uniformising.

Infinite area

Lang-Fang Wu introduced in [Wu93] two criteria to distinguish such immortal (i.e. infinite volume) cases.

Definition 5.3.4. Let g be a complete conformally flat metric on the plane $\mathbb{C}$. The **circumference at infinity** of g is

$$C_\infty(g) = \sup_{K \Subset \mathbb{C}} \inf \Big\{ \mathrm{L}_g(\partial U) : \text{open } U \supset \mathbb{C} \text{ with } U \supset K \Big\} \tag{5.6}$$

and its **aperature**

$$A(g) = \frac{1}{2\pi} \lim_{r\to\infty} \frac{\mathrm{L}_g\big(\partial \mathcal{B}_g(p; r)\big)}{r}. \tag{5.7}$$

Of the above examples we have: $C_\infty(g_{\mathbb{E}}) = \infty$ and $A(g_{\mathbb{E}}) = 1$, $C_\infty(g_\Sigma) = 2\pi$ and $A(g_\Sigma) = 0$. Since there is no natural choice (scaling) of the flat limit, we need the following slightly weaker notion of convergence.

[§] Two metrics g and h on a manifold $\mathcal{M}$ are *equivalent*, if $C^{-1}g \leq h \leq Cg$ for some $C \in (0, \infty)$.

Definition 5.3.5. A Ricci flow $\big(g(t)\big)_{t\in[0,\infty)}$ on a surface $\mathcal{M}^2$ has **modified subsequence convergence** if there exists a one-parameter family of diffeomorphisms $\big(\varphi_t\big)_{t\in[0,\infty)} \subset \mathrm{Diff}(\mathcal{M})$ such that for any sequence $\big(t_j\big)_{j\in\mathbb{N}}$ with $t_j \to \infty$ as $j \to \infty$, there exists a subsequence (denoted again by $\big(t_j\big)_{j\in\mathbb{N}}$) for which $\varphi_{t_j}^* g(t_j)$ converges uniformly locally as $j \to \infty$.

Theorem 5.3.6 (Part of [Wu93, Main Theorem])**.** *A complete Ricci flow $\big(\mathrm{e}^{2u(t)}|\mathrm{d}z|^2\big)_{t\in[0,\infty)}$ on $\mathbb{C}$ whose Gaussian curvature $\big\|\mathrm{K}_{u(0)}\big\|_{\mathrm{L}^\infty(\mathbb{C})} < \infty$ and the gradient of its conformal factor $\big\|\mathrm{grad}\big(\log u(0)\big)\big\|_{\mathrm{L}^\infty(\mathbb{C})} < \infty$ are bounded initially has modified subsequence convergence. Moreover, assuming that the Gaussian curvature $\mathrm{K}_{u(0)} > 0$ is initially positive, then the limiting metric is a cigar, if $C_\infty\big(u(0)\big) < \infty$, or flat if $A\big(u(0)\big) > 0$.*

Isenberg and Mohammad Javaheri proved a variant [IJ09] where they replaced the requirements of positive curvature and bounded gradient of the conformal factor by just imposing the conformal factor to be bounded initially or in other words, the initial metric to be *equivalent* to a flat metric.

Theorem 5.3.7 ([IJ09, Theorem 2])**.** *Let $\big(g(t)\big)_{t\in[0,\infty)}$ be a complete Ricci flow on the plane $\mathbb{C}$ such that $g(0)$ is equivalent to a flat metric and has bounded curvature $\sup_{\mathbb{C}}\big|\mathrm{K}_{g(0)}\big| < \infty$. Then it has smooth modified subsequence convergence to a flat metric.*

Schnürer, Schulze and Simon have investigated the stability of the Euclidean space under Ricci flow (mainly in higher dimensions), and state the following two results about the two-dimensional case in [SSS08], which generalise the above results.

Theorem 5.3.8 ([SSS08, Theorems A.1 and A.2])**.** *Let $\big(g(t)\big)_{t\in[0,\infty)}$ be a complete conformal Ricci flow on the plane $\big(\mathbb{C}, g_{\mathbb{E}}\big)$ such that $g(0)$ is equivalent to $g_{\mathbb{E}}$. Then it converges subsequentially, locally and smoothly to a flat metric (i.e. a multiple of $g_{\mathbb{E}}$). Moreover, if we write $g(t) = \mathrm{e}^{2u(t)} g_{\mathbb{E}}$ and additionally have*

$$\sup_{\mathbb{C}\setminus\mathbb{D}_r} \big|u(0)\big| \quad \overset{r\to\infty}{\longrightarrow} \quad 0, \tag{5.8}$$

then the convergence is global with unique limit $g_{\mathbb{E}}$, more precisely for each $k \in \mathbb{N}_0$ one has

$$\Big\|g(t) - g_{\mathbb{E}}\Big\|_{\mathrm{C}^k(\mathbb{C}, g_{\mathbb{E}})} \quad \overset{t\to\infty}{\longrightarrow} \quad 0.$$

Finite area

The case of finite area is even more subtle since the unnormalised (instantaneously complete) Ricci flow does not preserve the area, and in fact, it ends at the time $T = \frac{1}{4\pi}\mathrm{vol}_{g(0)}\,\mathbb{C}$. Daskalapoulos and Hamilton [DH04] showed that this finite time singularity is of type II, i.e. the blow-up of the curvature is faster than $\frac{1}{T-t}$.¶

Theorem 5.3.9 ([DH04, Theorem 1.2])**.** *Let $\big(g(t)\big)_{t\in[0,T)}$ be a conformally flat, complete Ricci flow on $\mathbb{C}$ such that $T = \frac{1}{4\pi}\mathrm{vol}_{g(0)}\,\mathbb{C} < \infty$ and $\sup_{\mathbb{C}}\big|\mathrm{K}_{g(t)}\big| < \infty$ for all $t \in [0, T)$. Then there exist constants $c > 0$ and $C < \infty$ such that for $t \in [0, T)$*

$$\frac{c}{(T-t)^2} \le \sup_{\mathbb{C}} \mathrm{K}_{g(t)} \le \frac{C}{(T-t)^2}.$$

¶See [Ham95b, §16] for a classification of Ricci flow singularities.

John R. King did a formal analysis of this extinction behaviour for radial symmetric solutions [Kin93]. He describes an *outer region* where the solution becomes asymptotically a hyperbolic cusp, and an *inner region* where the solution becomes asymptotically a cigar. Daskalapoulos verified this rigorously in collaboration with del Pino in the rotationally symmetric case [DdP07] and with Šešum in the general case [DS10]. The survey [IMS11, Theorem 4.4] summarises these results as follows:

Theorem 5.3.10 (Daskalapoulos, del Pino and Šešum. [DdP07], [DS10])**.** *Let $\big(v(t)|\mathrm{d}z|^2\big)_{t\in[0,T)}$ be an instantaneously complete Ricci flow on $\mathbb{C}$ such that $v(0)$ is compactly supported*[‖] *and $T = \frac{1}{4\pi}\int_{\mathbb{C}} v(0)\,\mathrm{d}\mu$.*

(i) In the outer region *where $\log|z| > \frac{T}{T-t}$, the Ricci flow $\big(v(t)|\mathrm{d}z|^2\big)_{t\in[0,T)}$ converges after an appropriate change of variables and rescaling to a hyperbolic cusp solution*

$$\frac{2t}{|z|^2(\log|z|)^2}|\mathrm{d}z|^2.$$

(ii) In the inner region *where $\log|z| \le \frac{T}{T-t}$, the Ricci flow $\big(v(t)|\mathrm{d}z|^2\big)_{t\in[0,T)}$ decays exponentially in time and converges, after another appropriate change of variables and rescaling to a cigar solution*

$$\left(\frac{T}{2}|z|^2 + \mathrm{e}^{\frac{2T}{T-t}}\right)^{-1}|\mathrm{d}z|^2.$$

‖Although $v(0)|\mathrm{d}z|^2$ is not a valid metric, it makes sense to regard $\big(v(t)|\mathrm{d}z|^2\big)_{t\in(0,T)}$ as a complete Ricci flow with $v(0)|\mathrm{d}z|^2$ as its initial condition.

Chapter 6

Ricci flows with unbounded curavature

The generalisation in the previous chapter (Theorem 5.1.2) of the classical existence results for Ricci flows on surfaces allows us to start from surfaces with unbounded curvature. Hence, one might wonder whether Ricci flows with unbounded curvature at later times does exist at all. The answer to this question might affect the proof of the full uniqueness Conjecture 5.2.1 for conformally hyperbolic solutions, since all our approaches so far use some kind of curvature bounds (→ §5.2.1). Note that all results in this chapter have been published in the article [GT12] by Peter Topping and the author.

6.1 Introduction

A Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ starting from an incomplete surface or a surface with unbounded curvature will have unbounded curvature in the sense

$$\sup_{t\in(0,T]}\sup_{\mathcal{M}}\big|\mathrm{K}_{g(t)}\big| = \infty,$$

as the curvature blows up in the limit $t \searrow 0$. So one might wonder:

> *Can there exist a complete Ricci flow $\big(g(t)\big)_{t\in[0,T]}$, on a surface, with unbounded curvature for some $t \in (0,T)$, i.e. $\sup_{\mathcal{M}}\big|\mathrm{K}_{g(t)}\big| = \infty$?*

Note that Chen's *a priori* estimate (Corollary 2.5.3) excludes the possibility of having an (instantaneously) complete Ricci flow with Gaussian curvature unbounded from below for any later time. However, in this chapter we are going to construct an example of a Ricci flow of unbounded curvature, confirming the above question.

Theorem 6.1.1 ([GT12, Theorem 1.2])**.** *On every non-compact Riemann surface $\mathcal{M}^2$ there exists a complete immortal Ricci flow $\big(g(t)\big)_{t\in[0,\infty)}$ with unbounded curvature $\sup_{\mathcal{M}}\mathrm{K}_{g(t)} = \infty$ for all $t \in [0,\infty)$.*

Remark 6.1.2. By taking the Cartesian product of the flows in Theorem 6.1.1 with $\mathbb{R}^n$, one can construct examples of complete Ricci flows with unbounded curvature in higher dimensions.

Note that, prior to our work in [GT12] Esther Cabezas-Rivas and Burkhard Wilking have shown us constructions in higher dimensions where one has Ricci flows with unbounded curvature for all times which also have positive curvature (→ [CRW11]).

Outline

We start by summarising some known facts about the cigar and its associated Ricci flow solution (§6.2.1). In the following main part (§6.2.2) we prove some results about Ricci flows starting at a surface containing a truncated cigar. The goal is to show that the cigar is mostly preserved for as long as we want regardless of the geometry of the surrounding surface — provided the cigar was initially long enough. The proof involves the construction of suitable barriers using the maximum principle in conjunction with a version of Perelman's pseudolocality theorem due to Chen (Theorem 2.6.1). Exploiting these barriers we will apply an isoperimetric inequality by Gerrit Bol (Theorem A.4.1) to show that the maximum of the curvature in that cigar region is universally bounded from below by a suitable positive constant for as long as we want. Finally, in Section 6.3 we conclude the proof of Theorem 6.1.1 by patching a sequence of longer and longer shrunken cigars onto an arbitrary non-compact Riemannian surface such that both its curvature is unbounded and if we flow it using Theorem 5.1.2 the results of Section 6.2 ensure that the curvature stays unbounded. We remark that the earlier construction of Cabezas-Rivas and Wilking [CRW11] in higher dimensions is also constructed out of cigars.

6.2 *A priori* estimates

6.2.1 Hamilton's cigar

In this section we collect some known results about Hamilton's cigar soliton. For further details we refer to [CK04, §2.2.1] or [CLN06, §4.3].

On $\mathbb{C}$ we write the cigar metric g_Σ in terms of a global complex coordinate $z = x + \mathrm{i}y$

$$g_\Sigma(z) = \frac{|\mathrm{d}z|^2}{1+|z|^2} \quad \text{with Gaussian curvature} \quad \mathrm{K}_{g_\Sigma}(z) = \frac{2}{1+|z|^2}. \tag{6.1}$$

The centric geodesic ball of radius $r > 0$ corresponds in complex coordinates to

$$\mathcal{B}_{g_\Sigma}(0; r) = \mathbb{D}_{\sinh r} \subset \mathbb{C}.$$

Being a steady Ricci soliton, one can associate a self-similar Ricci flow – the *cigar solution* $\big(g_\Sigma(t)\big)_{t\in\mathbb{R}}$ on $\mathbb{C}$ – defined by

$$g_\Sigma(t; z) = \mathrm{e}^{2u_\Sigma(t,z)}|\mathrm{d}z|^2 = \frac{|\mathrm{d}z|^2}{\mathrm{e}^{4t}+|z|^2} \quad \Longrightarrow \quad u_\Sigma(t,z) = -\frac{1}{2}\log\left(\mathrm{e}^{4t}+|z|^2\right). \tag{6.2}$$

After puncturing at its tip $z = 0$, we write the cigar in cylindrical coordinates $(\ell, \theta) \in \mathbb{R}\times\mathbb{S}^1 =: \mathcal{C}$ defined by $z = \mathrm{e}^{\ell+\mathrm{i}\theta}$ as

$$g_\Sigma(\ell,\theta) = \frac{\mathrm{d}\ell^2+\mathrm{d}\theta^2}{\mathrm{e}^{-2\ell}+1} \le \mathrm{d}\ell^2+\mathrm{d}\theta^2. \tag{6.3}$$

For large $|z| = \mathrm{e}^\ell$ the cigar is almost a flat cylinder and therefore we expect the geodesic distance from its tip at $\ell = -\infty$ to a point (ℓ,θ) to be roughly ℓ for large ℓ. In fact we have

$$\ell + \log 2 \le \mathrm{dist}_{g_\Sigma}\big((-\infty,\theta),(\ell,\theta)\big) = \operatorname{arsinh}\mathrm{e}^\ell \le \ell + 1 \qquad \text{for all } \ell \ge 0. \tag{6.4}$$

Similarly, the area of a ball of large radius $r > 0$ around the tip should be roughly the area of a cylinder of length $\ell \sim r$, i.e. $\mathrm{vol}_{g_\Sigma} \sim 2\pi\ell$; more precisely we have the lower estimates

$$\mathrm{vol}_{g_\Sigma}\big((-\infty,\ell)\times\mathbb{S}^1\big) = 2\pi\log\cosh\operatorname{arsinh}\mathrm{e}^{\ell} \geq 2\pi\ell \qquad \text{for all } \ell \geq 0 \tag{6.5}$$

$$\text{and} \qquad \mathrm{vol}_{g_\Sigma}\big(\mathcal{B}_{g_\Sigma}(0;r)\big) = 2\pi\log\cosh r \geq 2\pi(r-\log 2) \qquad \text{for all } r \geq 0. \tag{6.6}$$

The following two estimates quantify further the assertion above that the cigar is asymptotically a flat cylinder: For any $r > 0$ and for any point in $\mathbb{C}\setminus\mathcal{B}_{g_\Sigma}(0;r)$, i.e. for all $(\ell,\theta)\in\big(\log\sinh r,\infty\big)\times\mathbb{S}^1\subset\mathcal{C}$, we have the rough estimates for the metric

$$\left(1-\frac{1}{r^2}\right)\big(\mathrm{d}\ell^2+\mathrm{d}\theta^2\big) \leq \big(\tanh r\big)^2\big(\mathrm{d}\ell^2+\mathrm{d}\theta^2\big) \leq g_\Sigma(\ell,\theta) \leq \mathrm{d}\ell^2+\mathrm{d}\theta^2 \tag{6.7}$$

(because $\big(\tanh r\big)^2\big(\mathrm{d}\ell^2+\mathrm{d}\theta^2\big) = g_\Sigma(\log\sinh r,\theta)$) and for the Gaussian curvature

$$\sup_{(\log\sinh r,\infty)\times\mathbb{S}^1}\mathrm{K}_{g_\Sigma} = \frac{2}{(\cosh r)^2} \leq \frac{2}{r^2}. \tag{6.8}$$

In cylindrical coordinates the self-similarity of the cigar solution is more obvious: under the Ricci flow it just translates:

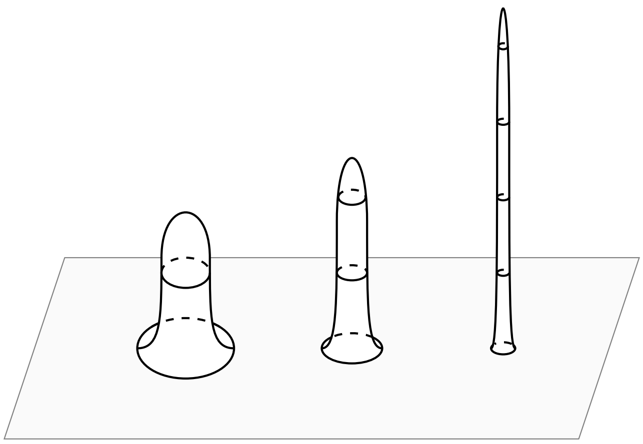
$$g_\Sigma(t;\ell,\theta) = \frac{\mathrm{d}\ell^2+\mathrm{d}\theta^2}{\mathrm{e}^{4t-2\ell}+1} = g_\Sigma(\ell-2t,\theta). \tag{6.9}$$

Consequently, thinking again of it as a cylinder for sufficiently large $\ell > 2t$ which now translates in time, we expect the distance to its tip to behave like $\sim \ell-2t$, and similarly the area of the g_Σ-geodesic ball, centred at the tip, of radius r should behave like the area of a cylinder of length $r-2t$, i.e. $\sim 2\pi(r-2t)$; more precisely we have the lower estimates

$$\mathrm{dist}_{g_\Sigma(t)}\big(0,\partial\mathcal{B}_{g_\Sigma}(0;r)\big) = \operatorname{arsinh}\mathrm{e}^{(\log\sinh r)-2t} \geq r-2t \qquad \text{and} \tag{6.10}$$

$$\mathrm{vol}_{g_\Sigma(t)}\mathcal{B}_{g_\Sigma}(0;r) = 2\pi\log\cosh\operatorname{arsinh}\mathrm{e}^{(\log\sinh r)-2t} \geq 2\pi\big(r-2t-\log 2\big). \tag{6.11}$$

6.2.2 Ricci flows containing cigars



The Ricci flow we construct to prove Theorem 6.1.1 will start with a metric containing countably many truncated cigars, at smaller and smaller scales, and which are longer and longer.

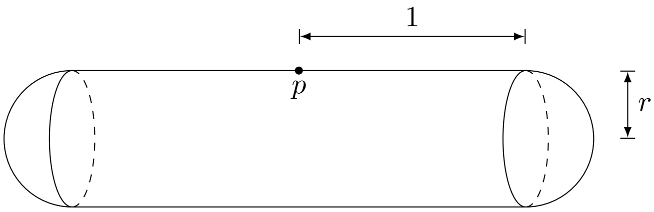
The proof will rely on each truncated cigar evolving much like a whole cigar would evolve, for a very long time, irrespective of how wild the metric is beyond the cigar part. However, great care is required to establish such behaviour, as is indicated by the following example.

Theorem 6.2.1 (Topping [Top05]). *Given $\varepsilon > 0$ (however small) there exists a Ricci flow $\big(g(t)\big)_{t\in[0,\varepsilon)}$ on the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ (smooth on $[0,\varepsilon)\times\overline{\mathbb{D}}$) such that $g(0)$ is the standard flat metric on the unit disc, but for which*

$$\mathrm{vol}_{g(t)}(\mathbb{D}) \to 0 \qquad \textit{and} \qquad \inf_{\mathbb{D}} \mathrm{K}_{g(t)} \to \infty$$

as $t \nearrow \varepsilon$. Moreover, if we write $g(t) = \mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ then $\sup_{\mathbb{D}} u(t) \to -\infty$ as $t \nearrow \varepsilon$.

Proof. Let $\mathcal{M}$ be $\mathbb{S}^2$, and equip it with an initial metric g_0 so that $(\mathcal{M}, g_0)$ arises by taking a cylinder $\mathbb{S}^1_r \times [-1,1]$, where $\mathbb{S}^1_r$ is the circle of length $2\pi r$, and capping the ends with round hemispheres of the appropriate size:



If we take the Ricci flow $g(t)$ starting at $g(0) = g_0$ given by Theorem 5.1.2 then we know it exists precisely until time $\frac{1}{8\pi}\mathrm{vol}_{g_0}\mathbb{S}^2 = \frac{1}{2}(r + r^2) = \varepsilon$ (for appropriate $r > 0$) and as $t \nearrow \varepsilon$, we have

$$\inf_{\mathcal{M}} \mathrm{K}_{g(t)} \to \infty,$$

because as a result of Corollary 3.4.6, $g(t)$ converges to the shrinking *round* sphere as it vanishes.

Now pick $p \in \mathcal{M}$ mid-way along the cylindrical part of $\mathbb{S}^2$. Then g_0 is flat on $\overline{\mathcal{B}_{g_0}(p;1)}$. Consider the map $F : \overline{\mathbb{D}} \to \mathbb{S}^2$ defined by restricting the exponential map $\exp_p$, with respect to the metric $g(0)$, where $\overline{\mathbb{D}}$ is seen as the closed unit 2-disc in the tangent space $\mathrm{T}\mathbb{S}^2_p$. This way, the image of F does not intersect the hemispherical caps in the construction of the initial manifold, and is an immersion.

We construct a Ricci flow $\big(\hat{g}(t)\big)_{t\in[0,\varepsilon)}$ on $\overline{\mathbb{D}}$ by defining $\hat{g}(t) = F^*\big(g(t)\big)$ at each time $t \in [0,\varepsilon)$. Then $\big(\overline{\mathbb{D}}, \hat{g}(0)\big)$ is the flat unit 2-disc, the curvature blows up everywhere at time $\varepsilon = \frac{1}{2}(r + r^2)$, and the area decreases to zero at this time. $\square$

Using an appropriate barrier argument, one can transfer the result to our situation where we have an embedded cigar.

Corollary 6.2.2. *Given $R > 0$ (however large) and $\varepsilon > 0$ (however small) there exists a Ricci flow $\big(g(t)\big)_{t\in[0,\varepsilon)}$ on $\mathbb{D}_{\sinh R} \subset \mathbb{C}$ such that $g(0) = g_\Sigma\big|_{\mathbb{D}_{\sinh R}}$ (a truncated cigar of arbitrarily long length R) but for which*

$$\mathrm{vol}_{g(t)}\big(\mathbb{D}_{\sinh R}\big) \longrightarrow 0 \qquad \textit{as } t \nearrow \varepsilon.$$

Proof. By virtue of Theorem 6.2.1 (and after an appropriate rescaling) we have a Ricci flow $\big(\mathrm{e}^{2\hat{u}(t)}|\mathrm{d}z|^2\big)_{t\in[0,\varepsilon)}$ on $\overline{\mathbb{D}_{\sinh R}}$ with $\hat{u}(0) \equiv 0$ and $\mathrm{vol}_{\hat{u}(t)}\big(\mathbb{D}_{\sinh R}\big) \to 0$ as $t \nearrow \varepsilon$. By standard parabolic existence theory, there exists a unique classical solution

$u : [0, \varepsilon) \times \overline{\mathbb{D}_{\sinh R}} \longrightarrow \mathbb{R}$ of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u = \mathrm{e}^{-2u} \Delta u & \text{on } (0, \varepsilon) \times \mathbb{D}_{\sinh R} \\ u = \hat{u} & \text{on } (0, \varepsilon) \times \partial \mathbb{D}_{\sinh R} \\ u = u_\Sigma \leq 0 & \text{on } \{0\} \times \mathbb{D}_{\sinh R}. \end{cases}$$

By direct comparison (Theorem 2.3.1) we see that $u(t) \leq \hat{u}(t)$ for all $t \in [0, \varepsilon)$ and consequently, if we write $g(t) = \mathrm{e}^{2u(t)} |\mathrm{d}z|^2$ on $\mathbb{D}_{\sinh R}$ for all $t \in [0, \varepsilon)$, then $\big(g(t)\big)_{t \in [0, \varepsilon)}$ is a smooth Ricci flow on $\mathbb{D}_{\sinh R}$ with $g(0) = g_\Sigma\big|_{\mathbb{D}_{\sinh R}}$ and $\mathrm{vol}_{g(t)}\big(\mathbb{D}_{\sinh R}\big) \leq \mathrm{vol}_{\hat{u}(t)}\big(\mathbb{D}_{\sinh R}\big) \longrightarrow 0$ as $t \nearrow \varepsilon$. $\square$

In other words, the Ricci flow can suck out an arbitrarily large amount of area from a remote part of the manifold, in an arbitrarily short period of time. Despite this extreme behaviour, the results of this section show that a truncated cigar evolving as part of a larger *complete* Ricci flow cannot suffer such a fate, and must evolve like a complete cigar for an arbitrarily long time.

Definition 6.2.3. A surface $\big(\mathcal{M}^2, g\big)$ contains at the point $p \in \mathcal{M}$ a cigar of length $R > 0$ at scale $\alpha > 0$ if $\mathcal{B}_g(p; R) \Subset \mathcal{M}$ and there exists an isometric diffeomorphism ψ from $\mathcal{B}_g(p; R)$ to the cigar at scale α (i.e. the rescaled cigar $\alpha^2 g_\Sigma$) restricted to the disc $\mathbb{D}_{\sinh(\alpha^{-1} R)} \subset \mathbb{C}$:

$$\psi : \Big(\mathcal{B}_g(p; R), g\Big) \longrightarrow \Big(\mathbb{D}_{\sinh(\alpha^{-1} R)}, \alpha^2 g_\Sigma\Big).$$

The next lemma obtains curvature control at a point part of the way down an evolving truncated cigar, at least while the point remains well within the interior of the evolving manifold and the tip does not get too close.

Lemma 6.2.4. *For some universal constant $B > 0$ and for all times $T > 0$, radii $r_0 \geq \sqrt{BT}$ and lengths $\tilde{R} \geq 4(r_0 + 1)$, if $\big(g(t)\big)_{t \in [0,T]}$ is a Ricci flow on a non-compact Riemann surface $\mathcal{M}^2$ such that $\big(\mathcal{M}, g(0)\big)$ contains at the point $p \in \mathcal{M}$ a cigar of length $2\tilde{R}$ at unit scale, then we have*

$$\sup_{\partial \mathcal{B}_{g(0)}(p; \tilde{R})} \mathrm{K}_{g(t)} \leq 2 r_0^{-2} \qquad \text{for all } t \in [0, \tau] \tag{6.12}$$

where

$$\tau := \sup \left\{ t_0 \in [0, T] : \begin{array}{l} \mathcal{B}_{g(t)}(q; r_0) \Subset \mathcal{M} \setminus \{p\} \\ \text{for all } q \in \partial \mathcal{B}_{g(0)}(p; \tilde{R}) \text{ and } t \in [0, t_0] \end{array} \right\} > 0. \tag{6.13}$$

The lemma exploits Chen's two-dimensional version (Theorem 2.6.1) of Perelman's pseudolocality theorem to gain arbitrary control of the curvature (r_0 can be chosen as large as we want). Since we are about to apply that theorem on a region where the cigar is almost a cylinder, the non-collapsed condition (iii) would cause trouble for large r_0. Therefore we will puncture the cigar at its tip and lift it to its universal cover (being locally a half plane) such that the problem vanishes and we can establish the curvature estimate for the lifted Ricci flow. The restriction to $[0, \tau]$ in (6.12) is basically the price to pay for that trick to still fulfil condition (i) of that theorem on the universal cover of the punctured surface $\mathcal{M} \setminus \{p\}$. However this turns out not to be a restriction because the barriers we construct in the next Lemma 6.2.5 will ensure that for sufficiently large $\tilde{R}$ we will have maximal $\tau = T$ if we additionally require the Ricci flow $g(t)$ to be instantaneously complete.

Proof. For $v_0 = \frac{3\pi}{4}$ we obtain a universal $B > 0$ as the constant $C = C(v_0) > 0$ in Theorem 2.6.1. Fix $T > 0$, $r_0 \geq \sqrt{BT}$ and $\tilde{R} \geq 4(r_0 + 1)$. Let $\big(g(t)\big)_{t\in[0,T]}$ be a Ricci flow such that $\big(\mathcal{M}, g(0)\big)$ contains at the point $p \in \mathcal{M}$ a cigar of length $2\tilde{R}$ at unit scale.

By virtue of the Uniformisation Theorem A.2.4, since $\mathcal{M}$ is non-compact its universal cover is conformally the plane $\mathbb{C}$ or the disc $\mathbb{D}$. Because it suffices to establish the curvature estimate (6.12) on any sheet of its universal cover, we can assume without loss of generality that $\mathcal{M}$ is conformal to $\mathbb{C}$ or $\mathbb{D}$ and in particular, punctured at the tip of the cigar, $\mathcal{M}_p := \mathcal{M} \setminus \{p\}$ is homeomorphic to a cylinder.

Using (possibly a second time) the universal covering $\pi : \widetilde{\mathcal{M}_p} \to \mathcal{M}_p$ to lift the Ricci flow $\tilde{g}(t) := \pi^* g(t)$ for all $t \in [0, T]$, we can write the lifted punctured cigar $\tilde{g}(0)$ on $\pi^{-1}\big(\mathcal{B}_{g(0)}(p; 2\tilde{R})\big)$ locally in 'lifted' cylindrical coordinates $(\ell, \tilde{\theta}) \in (-\infty, \log\sinh 2\tilde{R}) \times \mathbb{R}$ as

$$\tilde{g}(0; \ell, \theta) = \frac{\mathrm{d}\ell^2 + \mathrm{d}\tilde{\theta}^2}{\mathrm{e}^{-2\ell} + 1}.$$

Now fix any point $q \in \pi^{-1}\Big(\partial\mathcal{B}_{g(0)}(p; \tilde{R})\Big)$. We want to apply Theorem 2.6.1 to $\big(\tilde{g}(t)\big)_{t\in[0,\tau)}$ on $\mathcal{B}_{\tilde{g}(t)}(q; r_0)$. By definition of τ, we have $\mathcal{B}_{\tilde{g}(t)}(q; r_0) \Subset \widetilde{\mathcal{M}_p}$ for all $t \in [0, \tau)$, and therefore condition (i) of that theorem is fulfilled. As $\tilde{R} \geq 4(r_0 + 1)$ we certainly have $\mathcal{B}_{\tilde{g}(0)}(q; r_0) \subset \pi^{-1}\Big(\mathcal{B}_{g(0)}(p; 2\tilde{R}) \setminus \mathcal{B}_{g(0)}(p; \tilde{R}/2)\Big) \simeq \big(\log\sinh \tilde{R}/2, \log\sinh 2\tilde{R}\big) \times \mathbb{R}$ by equation (6.4), and condition (ii) follows using estimate (6.8)

$$\sup_{\mathcal{B}_{\tilde{g}(0)}(q;r_0)} \Big|\mathrm{K}_{\tilde{g}(0)}\Big| \leq \sup_{(\log\sinh \tilde{R}/2, \infty)\times\mathbb{S}^1} \mathrm{K}_{g_\Sigma} \leq 8\tilde{R}^{-2} \leq r_0^{-2}.$$

Similarly, transferring (6.7) to this case i.e.

$$\frac{3}{4}\big(\mathrm{d}\ell^2 + \mathrm{d}\tilde{\theta}^2\big) \leq \tilde{g}(0) \leq \mathrm{d}\ell^2 + \mathrm{d}\tilde{\theta}^2 \qquad \text{on } \mathcal{B}_{\tilde{g}(0)}(q; r_0)$$

we have $\mathcal{B}_{\mathrm{d}\ell^2+\mathrm{d}\tilde{\theta}^2}(q; r_0) \subset \mathcal{B}_{\tilde{g}(0)}(q; r_0)$ and condition (iii):

$$\operatorname{vol}_{\tilde{g}(0)} \mathcal{B}_{\tilde{g}(0)}(q; r_0) \geq \frac{3}{4} \operatorname{vol}_{\mathrm{d}\ell^2+\mathrm{d}\tilde{\theta}^2} \mathcal{B}_{\mathrm{d}\ell^2+\mathrm{d}\tilde{\theta}^2}(q; r_0) = \frac{3\pi}{4} r_0^2.$$

Applying Theorem 2.6.1 for every $q \in \pi^{-1}\Big(\partial\mathcal{B}_{g(0)}(p; \tilde{R})\Big)$ yields the desired curvature estimate for $\big(\tilde{g}(t)\big)_{t\in[0,\tau)}$ on $\pi^{-1}\Big(\partial\mathcal{B}_{g(0)}(p; \tilde{R})\Big)$, and therefore also for $\big(g(t)\big)_{t\in[0,\tau]}$ on $\partial\mathcal{B}_{g(0)}(p; \tilde{R})$ by continuity. $\square$

The curvature estimate of Lemma 6.2.4 can be transformed into a constraint on the conformal factor of $g(t)$ on $\partial\mathcal{B}_{g(0)}(p; \tilde{R})$ which will allow us to apply a maximum principle to establish local barriers ($\to$ (6.14)) for $t \in [0, \tau]$. Once we have the lower barrier we can utilise it to show that the Ricci flow does not shrink the tip of the cigar too fast; hence τ in Lemma 6.2.4 will have to be maximal ($\tau = T$) and we will have upper and lower barriers for all $t \in [0, T]$.

Lemma 6.2.5 (Barriers*). *For some universal constants $\beta > 1, A > 0$ and for all times $T > 0$ and lengths $\tilde{R} \geq A(T+1)$, if $\big(g(t)\big)_{t\in[0,T]}$ is an instantaneously complete Ricci flow on a non-compact Riemann surface $\mathcal{M}^2$ such that $\big(\mathcal{M}, g(0)\big)$ contains at the point $p \in \mathcal{M}$ a cigar of length $2\tilde{R}$ at unit scale, then there exist cigar solutions $\big(g_\pm(t)\big)_{t\in\mathbb{R}}$ at scales $\beta^{\pm 1}$ which are locally upper and lower barriers for $g(t)$ in the sense that on $\mathcal{B}_{g(0)}(p;\tilde{R})$*

$$g_-(t) := \psi^*\Big(\beta^{-2} g_\Sigma\big(\beta^2 t\big)\Big) \leq g(t) \leq \psi^*\Big(\beta^2 g_\Sigma\big(\beta^{-2}(t-T)\big)\Big) =: g_+(t) \tag{6.14}$$

for all $t \in [0,T]$ with ψ as in Definition 6.2.3.

Proof. Let B be the universal constant from Lemma 6.2.4, fix $T > 0$, and define $r_0 = \sqrt{BT}$ and $\beta = \mathrm{e}^{\frac{2}{B}} > 1$. Then we can find a universal $A > 0$ such that $A(T+1) \geq \max\big\{\beta(2\beta T + r_0 + 1), 4(r_0+1)\big\}$ and choose an arbitrary $\tilde{R} \geq A(T+1)$. Note that this way $\tilde{R}$ fulfils the requirement of Lemma 6.2.4. Finally define $\tau \in [0,T]$ according to (6.13).

Without loss of generality we can write $\psi_* g(t)\big|_{\mathcal{B}_{g(0)}(p;\tilde{R})} = \mathrm{e}^{2u(t)}|\mathrm{d}z|^2$ on $\overline{\mathbb{D}_{\sinh \tilde{R}}}$. Using (6.2) we obtain the conformal factors of the proposed barriers $\psi_*\big(g_\pm(t)\big) = \mathrm{e}^{2u_\pm(t)}|\mathrm{d}z|^2$ of (6.14)

$$u_-(t,z) = u_\Sigma\big(\beta^2 t, z\big) - \log\beta \qquad \text{and} \qquad u_+(t,z) = u_\Sigma\big(\beta^{-2}(t-T), z\big) + \log\beta.$$

To establish (6.14) we are going to exploit the standard parabolic maximum principle for solutions of (2.7) on $[0,\tau] \times \overline{\mathbb{D}_{\sinh\tilde{R}}}$: Since by assumption $u(0,z) = u_\Sigma(0,z)$ for all $z \in \mathbb{D}_{\sinh\tilde{R}}$, we have initially

$$u_-(0,z) = u_\Sigma(0,z) - \log\beta \leq u(0,z) \leq u_\Sigma(-\beta^{-2}T, z) + \log\beta = u_+(0,z)$$

for all $z \in \overline{\mathbb{D}_{\sinh\tilde{R}}}$. For the requirement on the boundary $[0,\tau] \times \partial\mathbb{D}_{\sinh\tilde{R}}$, observe that estimating (with Lemma 6.2.4) and integrating the evolution equation (2.7) for $u(t)$ yields for all $t \in [0,\tau]$

$$\left|\frac{\partial}{\partial t}u(t)\right| = \left|\mathrm{K}_{g(t)}\right| \leq \frac{2}{BT} \leq \frac{2}{B\tau} \quad \text{and thus} \quad \left|u(t) - u(0)\right| \leq \frac{2}{B} = \log\beta \quad \text{on } \partial\mathbb{D}_{\sinh\tilde{R}}.$$

Hence, we have for all $t \in [0,\tau]$ and $z \in \partial\mathbb{D}_{\sinh\tilde{R}}$

$$\begin{aligned} & u(t,z) \leq u(0,z) + \log\beta \leq u_\Sigma\big(\beta^{-2}(t-T), z\big) + \log\beta = u_+(t,z) \\ \text{and} \qquad & u(t,z) \geq u(0,z) - \log\beta \geq u_\Sigma\big(\beta^2 t, z\big) - \log\beta = u_-(t,z). \end{aligned}$$

That is sufficient to apply the maximum principle (Theorem 2.3.1) and conclude

$$u_-(t,z) \leq u(t,z) \leq u_+(t,z),$$

on $[0,\tau] \times \mathbb{D}_{\sinh\tilde{R}}$, which proves (6.14) for all $t \in [0,\tau]$ after pulling back the metrics via ψ.

Now we will use the lower barrier to show that in fact, with our choice of $\tilde{R} \geq A(T+1)$ we have maximal $\tau = T$: Since $\big(g(t)\big)_{t\in[0,T]}$ is instantaneously complete we

*This cigar sandwiching technique which we have introduced in [GT12] has since been applied in [Top11].

may rephrase the definition of τ in (6.13) to

$$\tau = \max\Big\{t_0 \in [0,T] : \operatorname{dist}_{g(t)}\Big(p, \partial\mathcal{B}_{g(0)}\big(p;\tilde{R}\big)\Big) \geq r_0 \text{ for all } t \in [0,t_0]\Big\}. \tag{6.13*}$$

Now assume that $\tau < T$. With our choice of $\tilde{R} \geq \beta(2\beta T + r_0 + 1)$ we have

$$\begin{aligned}
\operatorname{dist}_{g(t)}\big(p, \partial\mathcal{B}_{g(0)}(p;\tilde{R})\big) &\geq \operatorname{dist}_{g_-(t)}\big(p, \partial\mathcal{B}_{g(0)}(p;\tilde{R})\big) \\
&= \operatorname{dist}_{\beta^{-2}g_\Sigma(\beta^2 t)}\big(0, \partial\mathcal{B}_{g_\Sigma}(0;\tilde{R})\big) \\
&\geq \beta^{-1}\Big(\tilde{R} - 2\beta^2 t\Big) && \text{using (6.10)} \\
&\geq \beta^{-1}\tilde{R} - 2\beta T \geq r_0 + 1
\end{aligned}$$

for all $t \in [0,\tau]$, which contradicts the maximality of τ in (6.13*) since $t \mapsto g(t)$ is continuous. Hence $\tau = T$. □

6.2.3 Curvature bound

Lemma 6.2.5 tells us that if a complete Ricci flow contains a truncated cigar at unit scale at time 0, then it remains close to the evolving cigar for later times in a C^0 sense. If it were sufficiently close in a C^2 sense, then it would be clear that the supremum of the curvature would remain bounded below by a universal positive constant, because this is true for the evolving cigar. By exploiting an isoperimetric inequality, the following proposition shows that this lower curvature bound will hold anyway.

Proposition 6.2.6. *For some universal $\varepsilon > 0$ and $A > 0$, and for all times $T > 0$ and lengths $R \geq A(T+1)$, if $\big(g(t)\big)_{t\in[0,T]}$ is an instantaneously complete Ricci flow on a non-compact Riemann surface $\mathcal{M}^2$ such that $\big(\mathcal{M}, g(0)\big)$ contains a cigar of length R at unit scale, then*

$$\sup_{\mathcal{M}} \mathrm{K}_{g(t)} \geq \varepsilon \qquad \textit{for all } t \in [0,T]. \tag{6.15}$$

Proof. Let $\beta > 1$ be the universal scaling for the barriers $g_\pm(t)$ from Lemma 6.2.5, and let $\tilde{A} > 0$ be the A from that same lemma. Fix a larger $A \geq 2\tilde{A}$ such that $A(T+1) \geq 2\beta^2(2T + \beta^2 + 1)$ for all $T > 0$. Now fix $T > 0$ and pick $R \geq A(T+1)$. For improved legibility we are going to abbreviate $B_r := \mathcal{B}_{g(0)}(p;r)$ for any $r > 0$. To establish (6.15) we want to apply an isoperimetric inequality due to Bol (Theorem A.4.1) to the domain $B_{\rho(t)}$ of area $2\pi\beta^2$ around the tip of the cigar where

$$\rho(t) := \max\Big\{r > 0 : \operatorname{vol}_{g(t)} B_r \leq 2\pi\beta^2\Big\}.$$

To see that $\rho(t) \leq \frac{R}{2}$ for all $t \in [0,T]$, i.e. the domain $B_{\rho(t)}$ stays in the region $B_{\frac{R}{2}}$ where Lemma 6.2.5 provides the barriers $g_\pm(t)$, we can use the lower barrier $g_-(t)$ to estimate the area with respect to $g(t)$ from below: Keeping in mind $\frac{R}{2\beta^2} \geq 2T + \beta^2 + 1$ we estimate

$$\begin{aligned}
\operatorname{vol}_{g(t)} B_{\frac{R}{2}} &\geq \operatorname{vol}_{g_-(t)} B_{\frac{R}{2}} = \operatorname{vol}_{\beta^{-2}g_\Sigma(\beta^2 t)} \mathcal{B}_{g_\Sigma}(0; R/2) \\
&\geq 2\pi\beta^{-2}\left(R/2 - 2\beta^2 T - \log 2\right) && \text{using (6.11)} \\
&\geq 2\pi\beta^2
\end{aligned}$$

for all $t \in [0,T]$. Meanwhile, the upper barrier allows us to estimate the length of the boundary $\partial B_{\rho(t)}$ independently of $t \in [0,T]$ by the circumference of the scaled cylinder

$$\mathrm{L}_{g(t)}\,\partial B_{\rho(t)} \le \mathrm{L}_{g_+(t)}\,\partial B_{\rho(t)} \le \mathrm{L}_{\beta^2 g_\Sigma} \partial \mathcal{B}_{g_\Sigma}\big(0;\rho(t)\big) \le 2\pi\beta$$

using (6.7). Now we can apply Bol's isoperimetric inequality, Theorem A.4.1, to conclude

$$\begin{aligned}\sup_{\mathcal{M}} \mathrm{K}_{g(t)} &\ge \sup_{B_{\rho(t)}} \mathrm{K}_{g(t)} \\ &\ge \frac{4\pi}{\mathrm{vol}_{g(t)}\, B_{\rho(t)}} - \frac{\big(\mathrm{L}_{g(t)}\,\partial B_{\rho(t)}\big)^2}{\big(\mathrm{vol}_{g(t)}\, B_{\rho(t)}\big)^2} \\ &\ge \frac{4\pi}{2\pi\beta^2} - \frac{(2\pi\beta)^2}{(2\pi\beta^2)^2} = \beta^{-2} =: \varepsilon. \end{aligned}$$

□

Corollary 6.2.7. *For some universal $\varepsilon > 0$ and $A > 0$, and for all times $T > 0$, scales $\alpha \in (0,1)$ and lengths $R \ge A\alpha^{-1}(T+1)$, if $\big(g(t)\big)_{t\in[0,T]}$ is an instantaneously complete Ricci flow on a non-compact Riemann surface $\mathcal{M}^2$ such that $\big(\mathcal{M}, g(0)\big)$ contains a cigar of length R at scale α, then*

$$\sup_{\mathcal{M}} \mathrm{K}_{g(t)} \ge \varepsilon\alpha^{-2} \qquad \textit{for all } t \in [0,T]. \tag{6.16}$$

Proof. Fix $T > 0$ and $\alpha \in (0,1)$. From the preceding Proposition 6.2.6 we obtain the universal constants $\varepsilon > 0$ and $A > 0$. Define $\bar{T} := \alpha^{-2}T$ and observe that with $R \ge A\alpha^{-1}(T+1) \ge A\alpha(\alpha^{-2}T+1) = \alpha A(\bar{T}+1)$ we have $\bar{R} := \alpha^{-1}R \ge A(\bar{T}+1)$ as required in Proposition 6.2.6. Let $g(t)$ be an instantaneously complete Ricci flow such that $\big(\mathcal{M}, g(0)\big)$ contains a cigar of length R at scale α. Then $\big(\bar{g}(s)\big)_{s\in[0,\alpha^{-2}T]}$ with $\bar{g}(s) = \alpha^{-2}g(\alpha^2 s)$ is another instantaneously complete Ricci flow such that $\big(\mathcal{M}, \bar{g}(0)\big)$ contains a cigar of length $\bar{R} = \alpha^{-1}R$ at unit scale and we may apply Proposition 6.2.6 to conclude

$$\sup_{\mathcal{M}} \mathrm{K}_{g(t)} = \sup_{\mathcal{M}} \mathrm{K}_{\alpha^2\bar{g}(\alpha^{-2}t)} = \sup_{\mathcal{M}} \alpha^{-2}\mathrm{K}_{\bar{g}(\alpha^{-2}t)} \ge \alpha^{-2}\varepsilon$$

for all $t \in [0,T]$. □

6.3 Ricci flow of unbounded curvature

In this section we give a proof of Theorem 6.1.1. It is clearly sufficient to find an instantaneously complete Ricci flow rather than a complete one since we can always adjust the flow a little in time (i.e. consider $g(t+\varepsilon)$).

The strategy is to construct an appropriate $g(0)$ containing lots of cigars at different scales and of different lengths, then to flow the metric using Theorem 5.1.2 and apply the results of the previous section, and in particular Corollary 6.2.7, to show that the curvature is unbounded.

Proof of **Theorem 6.1.1**. Let $\bar{g}_0$ be any conformal metric on $\mathcal{M}$. Since $\mathcal{M}$ is a non-compact Riemann surface, there exists a sequence of pairwise disjoint, simply connected, open subsets $\big(\mathcal{U}_k\big)_{k\in\mathbb{N}} \subset \mathcal{M}$ moving off to infinity in the sense that for all $\Omega \Subset \mathcal{M}$, the sets $\mathcal{U}_k$ are disjoint from Ω for sufficiently large k. For the sequences of scales $\alpha_k := \frac{1}{k}$ and times $T_k := k$, Corollary 6.2.7 permits us to choose a monotonically increasing sequence of lengths $R_k = Ak(k+1) \sim k^2$ compatible with α_k and

T_k. By virtue of the uniformisation theorem, for each $k \in \mathbb{N}$ there exists a conformal diffeomorphism

$$\psi_k : \mathcal{U}_k \longrightarrow \mathbb{D}_{2\sinh(\alpha_k^{-1} R_k)} \subset \mathbb{C}$$

onto the complex disc of radius $2\sinh\big(\alpha_k^{-1} R_k\big)$. Abbreviate

$$\mathcal{V}_k := \psi_k^{-1}\left(\mathbb{D}_{\sinh(\alpha_k^{-1} R_k)}\right) \Subset \mathcal{U}_k$$

and choose smooth cut-off functions $f_k \in \mathrm{C}^\infty\big(\mathcal{M}, [0,1]\big)$ such that $\operatorname{spt} f_k \subset \mathcal{U}_k$ and $f_k\big|_{\mathcal{V}_k} \equiv 1$. We can then define a new smooth metric g_0 on $\mathcal{M}$ by

$$g_0 := \left(1 - \sum_{k\in\mathbb{N}} f_k\right) \bar{g}_0 + \sum_{k\in\mathbb{N}} f_k \psi_k^* \left(\alpha_k^2 g_\Sigma\right).$$

The surface $\big(\mathcal{M}, g_0\big)$ has infinite area since the truncated cigars we are pasting in each have area bounded below by some uniform positive constant (in fact their area is converging to infinity):

$$\begin{aligned} \operatorname{vol}_{g_0} \mathcal{M} &\geq \sum_{k\in\mathbb{N}} \operatorname{vol}_{g_0} \mathcal{V}_k = \sum_{k\in\mathbb{N}} \alpha_k^2 \operatorname{vol}_{g_\Sigma}\big(\mathcal{B}_{g_\Sigma}(0; \alpha_k^{-1} R_k)\big) \\ &\geq 2\pi \sum_{k\in\mathbb{N}} \alpha_k \big(R_k - \alpha_k\big) \geq 2\pi \sum_{k\in\mathbb{N}} \frac{R_1 - \frac{1}{k}}{k} = \infty, \qquad \text{using (6.6)} \end{aligned}$$

and so Theorem 5.1.2 provides an immortal, instantaneously complete Ricci flow $\big(g(t)\big)_{t\in[0,\infty)}$ with $g(0) = g_0$. To see that the curvature of $g(t)$ is unbounded at an arbitrary time $t \in [0,\infty)$, note that for any $k \in \mathbb{N}$ with $k > t$, we have $t \in [0, T_k]$, so by applying Corollary 6.2.7 to the flow, considering the k^{th} cigar in our construction, we find that

$$\sup_{\mathcal{M}} \mathrm{K}_{g(t)} \geq \varepsilon \alpha_k^{-2} = \varepsilon k^2,$$

and by letting $k \to \infty$ we find that $\sup_{\mathcal{M}} \mathrm{K}_{g(t)} = \infty$ as claimed. $\square$

Appendix A

Riemannian surfaces

A.1 Theorem of Gauß-Bonnet and variants

Theorem A.1.1 (Carl Friedrich Gauss (1827), Pierre Ossian Bonnet (1848); [Cha06, Theorem V.2.7]). *Let $\big(\mathcal{M}^2, g\big)$ be a compact, orientable Riemannian surface with smooth boundary $\partial\mathcal{M}$. Then*

$$\int_{\mathcal{M}} \mathrm{K}_g \,\mathrm{d}\mu_g + \int_{\partial\mathcal{M}} \kappa_g \,\mathrm{d}s = 2\pi\,\chi\big(\mathcal{M}\big). \tag{A.1}$$

Theorem A.1.2 (Alfred Huber [Hub57, Theorem 12]). *Let $\big(\mathcal{M}^2, g\big)$ be an open, finitely connected, orientable and complete Riemannian surface with finite area* $\mathrm{vol}_g\,\mathcal{M} < \infty$. *If $[\mathrm{K}_g]_+ \in \mathrm{L}^1(\mathcal{M}, g)$ or $[\mathrm{K}_g]_- \in \mathrm{L}^1(\mathcal{M}, g)$, then*

$$\int_{\mathcal{M}} \mathrm{K}_g \,\mathrm{d}\mu_g = 2\pi\,\chi\big(\mathcal{M}\big). \tag{A.2}$$

See also [SST03, Theorem 2.2.4].

A.2 Covering spaces and uniformisation

Definition A.2.1. Let $\pi : \widetilde{X} \to X$ be a continuous map between topological spaces X and $\widetilde{X}$. A **covering** $\pi : \widetilde{X} \to X$ is the triple consisting of the **covering map** π, the **cover** $\widetilde{X}$ and the **base** X. The map π is a **covering map**, if for every $x \in X$ there is an open neighbourhood $U \subset X$ such that its preimage $\pi^{-1}(U)$ is a disjoint union of open sets $\widetilde{U}_i \subset \widetilde{X}$, $i \in I$

$$\pi^{-1}(U) = \bigsqcup_{i\in I} \widetilde{U}_i$$

and the restriction to each connected component $\pi\big|_{\widetilde{U}_i} : \widetilde{U}_i \to U$ is a homeomorphism. The **covering group** is the group of **deck transformations**

$$\mathrm{Deck}(\pi) := \Big\{\varphi : \widetilde{X} \to \widetilde{X} \;\Big|\; \varphi \text{ is a homeomorphism with } \pi \circ \varphi = \pi\Big\}.$$

Finally, a covering is **universal** if $\widetilde{X}$ is simply connected.

Proposition A.2.2 (Existence of a universal cover). *If X is a connected and locally simply connected topological space, then there exist a simply connected topological space $\widetilde{X}$ and a covering map $\pi : \widetilde{X} \to X$. If $\hat{\pi} : \widehat{X} \to X$ is any other simply connected* *covering of X, then there is a homeomorphism $\phi : \widetilde{X} \to \widehat{X}$ such that $\hat{\pi} \circ \phi = \pi$. Moreover, the covering group is isomorphic to the fundamental group of the base X,* $\mathrm{Deck}(\pi) \cong \pi_1(X)$.

For a proof we refer for instance to [Lee00, Theorem 12.8 and 12.19]. It is easy to see that in the context of Riemannian manifolds one can pull-back the geometry of the base manifold to its cover. Hence, we may rephrase Proposition A.2.2 for Riemannian manifolds:

$$\begin{tikzcd}
& \big(\widetilde{\mathcal{M}}, \widetilde{g}\big) \arrow[dl, "\pi"'] \arrow[d, "\widetilde{\pi}"] \\
\big(\mathcal{M}, g\big) \arrow[r, "\phi", "\cong"'] & \big(\widetilde{\mathcal{M}}/\Gamma, \widetilde{\pi}_* \widetilde{g}\big)
\end{tikzcd}$$

Corollary A.2.3 (Universal cover of a Riemannian manifold)**.** *If $\big(\mathcal{M}^n, g\big)$ is a Riemannian manifold, then there exists a universal covering $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ such that*

(i) $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ is a simply connected Riemannian manifold with $\widetilde{g} := \pi^* g$;

(ii) The deck transformations $\mathrm{Deck}(\pi) < \mathrm{Isom}\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ are a subgroup of the freely and discretely acting isometries on $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$;

(iii) Abbreviating $\Gamma = \mathrm{Deck}(\pi)$ and writing $\widetilde{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}/\Gamma$ for the canonical projection, there is a global isometry

$$\phi : \big(\widetilde{\mathcal{M}}/\Gamma, \widetilde{\pi}_* \widetilde{g}\big) \longrightarrow \big(\mathcal{M}, g\big).$$

If $\widehat{\pi} : \widehat{\mathcal{M}} \to \mathcal{M}$ is another universal covering of $\mathcal{M}$, then there exists a global isometry $\phi : \big(\widetilde{\mathcal{M}}, \pi^* g\big) \to \big(\widehat{\mathcal{M}}, \widehat{\pi}^* g\big)$ such that $\widehat{\pi} \circ \phi = \pi$.

Uniformisation

The *uniformisation theorem* provides a classification of the geometry of surfaces to be either spherical, Euclidean or hyperbolic. Whilst for compact surfaces the Gauss-Bonnet Theorem A.1.1 relates the geometry to the topology via the sign of the Euler characteristic, for non-compact surfaces this is not true anymore; then again one requires additionally the conformal structure in order to determine the geometry.

Theorem A.2.4 (Uniformisation Theorem. Henri Poincaré [Poi07], Paul Koebe [Koe07])**.** *Let $\mathcal{M}^2$ be a simply connected Riemann surface. Then $\mathcal{M}$ is biholomorphic either to the sphere $\mathbb{S}^2$, to the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$. Moreover, every complete metric g on $\mathcal{M}$ is conformally equivalent either to $g_{\mathbb{S}}$ or $g_{\mathbb{E}}$ or $g_{\mathbb{H}}$ respectively.*

Using the universal covering one can uniquely relate any surface to a simply connected one, and therefore rephrase the uniformisation theorem for surfaces of non-trivial topology using Corollary A.2.3 about the existence of a universal covering.

Corollary A.2.5. *Every Riemannian surface $\big(\mathcal{M}^2, g\big)$ is conformally equivalent to a quotient of either $\big(\mathbb{S}^2, g_{\mathbb{S}}\big)$ or $\big(\mathbb{E}^2, g_{\mathbb{E}}\big)$ or $\big(\mathbb{H}^2, g_{\mathbb{H}}\big)$ by a discrete group of isometries which is isomorphic to the fundamental group $\pi_1(\mathcal{M})$. In particular, $\big(\mathcal{M}, g\big)$ admits a conformally equivalent and complete metric $g_{\bar{\kappa}}$ of constant Gaussian curvature $\mathrm{K}_{g_{\bar{\kappa}}} \equiv \bar{\kappa} \in \{1, 0, -1\}$.*

A.3 Schwarz lemma of Yau

Theorem A.3.1 (Shing-Tung Yau [Yau73]). *Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian surfaces such that for some constants $a_1 \geq 0$ and $a_2 > 0$ we have*

(i) $(\mathcal{M}_1, g_1)$ *is complete;* *(ii)* $\mathrm{K}_{g_1} \geq -a_1$; *(iii)* $\mathrm{K}_{g_2} \leq -a_2 < 0$.

Then any conformal map $\varphi : \big(\mathcal{M}_1, g_1\big) \longrightarrow \big(\mathcal{M}_2, g_2\big)$ *satisfies*

$$\varphi^*(g_2) \leq \frac{a_1}{a_2} g_1.$$

For convenience, we are going to present the proof. It uses the following generalised maximum principle by Hideki Omori, whose proof was simplified by Yau in [Yau75, Theorem 1, p. 206]:

Theorem A.3.2 (Omori [Omo67, Theorem A′, p. 211]). *On a complete Riemannian surface $(\mathcal{M}, g)$ with Gaussian curvature bounded from below, let $f \in \mathrm{C}^2(\mathcal{M})$ be bounded above. Then, for an arbitrarily point $p \in \mathcal{M}$ and for any $\varepsilon > 0$, there exists a point q depending on p such that*

(i) $\Delta_g f(q) < \varepsilon$; *(ii)* $\big|\mathrm{grad}_g f(q)\big|_g < \varepsilon$; *(iii)* $f(q) \geq f(p)$.

The essential idea [Yau75] to find q is to imagine the point $\big(q, f(q)\big)$ on the graph of f in $\mathcal{M} \times \mathbb{R}$ which is closest to the point (p, k) for some enormous $k \gg 1$.

Proof of **Theorem A.3.1**. By dilating g_1 and g_2, we may assume $a_1 = 1 = a_2$, that is $\mathrm{K}_{g_1} \geq -1 \geq \mathrm{K}_{g_2}$. Since we only need the theorem in the case that φ is strictly conformal, we will assume this in the proof and omit the minor adjustments required for the full theorem.* Define $w \in \mathrm{C}^\infty(\mathcal{M}_1)$ by

$$\varphi^*(g_2) = \mathrm{e}^{2w} g_1. \tag{A.3}$$

It remains to show that $w \leq 0$. Assume instead that there exists $p \in \mathcal{M}_1$ with $w(p) > 0$. Then we can choose an $\varepsilon \in (0, 1)$ such that

$$\varepsilon < \frac{\mathrm{e}^{w(p)} - \mathrm{e}^{-w(p)}}{1 + \mathrm{e}^{w(p)}}. \tag{A.4}$$

Now define $\tilde{w}(x) := -\mathrm{e}^{-w(x)}$ for all $x \in \mathcal{M}_1$. Since $(\mathcal{M}_1, g_1)$ is complete with curvature bounded from below and $\tilde{w}$ is bounded above, we may apply Theorem A.3.2 to find a point $q \in \mathcal{M}_1$ with

$$\Delta_{g_1} \tilde{w}(q) < \varepsilon, \qquad \big|\nabla \tilde{w}\big|^2_{g_1}(q) < \varepsilon \quad \text{and} \quad \tilde{w}(q) \geq \tilde{w}(p). \tag{A.5}$$

Since $x \mapsto -\mathrm{e}^{-x}$ is strictly increasing, we also have $w(q) \geq w(p) > 0$. Now compute

$$\begin{aligned} \Delta_{g_1} \tilde{w} &= \mathrm{e}^{-w} \Delta_{g_1} w - \mathrm{e}^{-w} |\nabla w|^2_{g_1} \\ &= \mathrm{e}^{-w} \Delta_{g_1} w - \mathrm{e}^{w} |\nabla \tilde{w}|^2_{g_1}. \end{aligned} \tag{A.6}$$

*In the weakly conformal case φ might either be constant (nothing to prove) or have isolated singular points $P := \{p_1, p_2, \ldots\}$. The function w we define in (A.3) will then have logarithmic singularities on P, but will be strictly negative close to such singularities and the $\tilde{w}$ of the proof could be adjusted to a smooth function on the whole of $\mathcal{M}_1$ (including P) without altering anything where w is positive.

By Lemma 2.2.1 we have

$$\Delta_{g_1} w = -\mathrm{e}^{2w}\,(\mathrm{K}_{g_2} \circ \varphi) + \mathrm{K}_{g_1},$$

which together with the curvature estimates $-\mathrm{K}_{g_2} \geq 1$ and $\mathrm{K}_{g_1} \geq -1$ gives $\mathrm{e}^{-w}\Delta_{g_1} w \geq \mathrm{e}^{w} - \mathrm{e}^{-w}$, and so (A.6) improves to

$$\Delta_{g_1} \tilde{w} \geq \mathrm{e}^{w}(1 - |\nabla \tilde{w}|^2_{g_1}) - \mathrm{e}^{-w}.$$

Evaluating at q, using (A.5) and the fact that $w(q) \geq w(p)$, we obtain

$$\varepsilon > \Delta_{g_1} \tilde{w}(q) \geq \mathrm{e}^{w(q)}(1-\varepsilon) - \mathrm{e}^{-w(q)} \geq \mathrm{e}^{w(p)}(1-\varepsilon) - \mathrm{e}^{-w(p)}$$

and hence

$$\varepsilon > \frac{\mathrm{e}^{w(p)} - \mathrm{e}^{-w(p)}}{1 + \mathrm{e}^{w(p)}}$$

which contradicts (A.4). □

Corollary A.3.3. *Let g be a complete and conformal metric on a hyperbolic surface $(\mathcal{M}^2, g_{\mathbb{H}})$. If $\mathrm{K}_g \geq -\kappa$ for some $\kappa \in [0,\infty)$, then*

$$g_{\mathbb{H}} \leq \kappa\, g. \tag{A.7}$$

A.4 Isoperimetric inequality

The following isoperimetric inequality due to Gerrit Bol allows us to estimate the maximum of the curvature on a surface's domain from below if we know its area and the length of its boundary. For an alternative proof using curvature flows, and further generalisations see [Top98] and [Top99].

Theorem A.4.1 (Bol [Bol41, Eq. (30) on p. 230])**.** *Let $\mathcal{U}$ be a simply-connected domain on a surface $(\mathcal{M}^2, g)$, then*

$$\big(\mathrm{L}_g\, \partial\mathcal{U}\big)^2 \geq 4\pi\, \mathrm{vol}_g(\mathcal{U}) - \big(\mathrm{vol}_g \mathcal{U}\big)^2 \sup_{\mathcal{U}} \mathrm{K}_g. \tag{A.8}$$

Appendix B

Parabolic partial differential equations

Let $\mathcal{M} \subseteqq \mathbb{R}^n$ be a domain with (possibly empty) boundary $\partial\mathcal{M}$ and let $\mathcal{T} \subset [0, \infty)$ be some time interval including 0. We define the parabolic distance on $\mathcal{T} \times \mathcal{M}$ by

$$\operatorname{dist}\Big((t_1, p_1), (t_2, p_2)\Big) := \sqrt{|t_1 - t_2|} + \big|\boldsymbol{p}_1 - \boldsymbol{p}_2\big| \tag{B.1}$$

for any space-time points $(t_1, p_1), (t_2, p_2) \in \mathcal{T} \times \mathcal{M}$. We are going to abuse notation and denote the parabolic boundary by

$$\partial(\mathcal{T} \times \mathcal{M}) := \big(\{0\} \times \mathcal{M}\big) \cup \big(\mathcal{T} \times \partial\mathcal{M}\big). \tag{B.2}$$

B.1 Hölder estimates for bounded solutions

The following theorem shows that a bounded solution of a (quasi-linear) parabolic partial differential equation is already Hölder continuous away from the parabolic boundary. For the sake of simplicity we state a weaker variant than [LSU68] which is sufficient for our applications.

Theorem B.1.1 (Simplified variant of [LSU68, Theorem V.1.1]). *For some time interval $\mathcal{T} \subset [0, \infty)$ including 0, some domain $\mathcal{M} \Subset \mathbb{R}^n$ and monotonically increasing functions $\lambda^{-1}, \Lambda, \beta \in \mathrm{C}\big([0, \infty), (0, \infty)\big)$ let $u \in \mathrm{C}^{1,2}\big(\mathcal{T} \times \mathcal{M}\big)$ be a solution of the quasi-linear parabolic equation in divergence form*

$$\frac{\partial}{\partial t} u(t, p) = \operatorname{div} \boldsymbol{A}\Big(t, p; u(t, p), \mathrm{D}u(t, p)\Big) + B\Big(t, p; u(t, p), \mathrm{D}u(t, p)\Big) \tag{B.3}$$

such that for all $(t, p) \in \mathcal{T} \times \mathcal{M}$, $w \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$

(i) $\boldsymbol{A} \in \mathrm{C}\big(\mathcal{T} \times \mathcal{M} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n\big)$ *satisfies the parabolicity conditions*

$$\lambda\big(|w|\big)\, |\boldsymbol{\xi}|^2 \le \big\langle \boldsymbol{A}(t, p; w, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle \quad \textit{and} \quad \big|\boldsymbol{A}(t, p; w, \boldsymbol{\xi})\big| \le \Lambda\big(|w|\big)\, |\boldsymbol{\xi}|; \tag{B.4}$$

(ii) $B \in \mathrm{C}\big(\mathcal{T} \times \mathcal{M} \times \mathbb{R} \times \mathbb{R}^n\big)$ *satisfies*

$$\big|B(t, p; w, \boldsymbol{\xi})\big| \le \beta\big(|w|\big)\big|\boldsymbol{\xi}\big|^2. \tag{B.5}$$

For $M := \big\|u\big\|_{\mathrm{L}^\infty(\mathcal{T} \times \mathcal{M})} < \infty$ and any $\mathcal{Q} \Subset \mathcal{T} \times \mathcal{M}$ with $\delta := \operatorname{dist}\big(\mathcal{Q}, \partial(\mathcal{T} \times \mathcal{M})\big) > 0$

there exists a constant $\alpha = \alpha\left(n, \lambda(M), \Lambda(M), \frac{M\beta(M)}{\lambda(M)}\right) \in (0,1]$ *such that*

$$\|u\|_{\mathrm{C}^{\alpha/2,\alpha}(\mathcal{Q})} \leq C = C\Big(n, M, \lambda(M), \Lambda(M), \beta(M), \delta\Big) < \infty. \tag{B.6}$$

In Section 4.3 we are going to apply this theorem to the normalised Ricci flow equation for the metric's conformal factor in the conformally hyperbolic case

$$\frac{\partial}{\partial t}\bar{u} = \operatorname{div}\big(\mathrm{e}^{-2\bar{u}}\,\mathrm{D}\bar{u}\big) + 2\mathrm{e}^{-2\bar{u}}\left|\mathrm{D}\bar{u}\right|^2 - 1. \tag{4.16}$$

In this setting we have $\boldsymbol{A}(t,p;w,\boldsymbol{\xi}) = \mathrm{e}^{-2w}\boldsymbol{\xi}$ and $B(t,p;w,\boldsymbol{\xi}) = 2\mathrm{e}^{-2w}|\boldsymbol{\xi}|^2 - 1$, therefore conditions (i) and (ii) are satisfied choosing $\lambda(w) = \mathrm{e}^{-2w}$ and $\Lambda(w) = \beta(w) = \mathrm{e}^{2w}$.

B.2 Parabolic Schauder estimates

Theorem B.2.1 (Simplified variant of [LSU68, Theorem IV.10.1])**.** *For some* $\alpha \in (0,1]$, $\lambda > 0$, $k,l \in \mathbb{N}_0$, *some time interval* $\mathcal{T} \subset [0,\infty)$ *and domain* $\mathcal{M} \subseteq \mathbb{R}^n$ *let* $u \in \mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}\big(\mathcal{T}\times\mathcal{M}\big)$ *be a solution of the linear parabolic equation*

$$\frac{\partial}{\partial t}u(t,p) = \Big\langle a(t,p), \operatorname{Hess} u(t,p)\Big\rangle + b(t,p) \tag{B.7}$$

such that $a \in \mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M}, \operatorname{Sym}_2\mathbb{R}^n)$, $b \in \mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M})$ *and for all* $(t,p) \in \mathcal{T}\times\mathcal{M}$ *we have the uniform parabolicity condition*

$$0 < \lambda|\boldsymbol{\eta}|^2 \leq \Big\langle a\big(t,p\big), \boldsymbol{\eta}\otimes\boldsymbol{\eta}\Big\rangle \qquad \textit{for all } \boldsymbol{\eta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{B.8}$$

For any $\mathcal{Q} \Subset \mathcal{T}\times\mathcal{M}$ *with* $\delta := \operatorname{dist}\big(\mathcal{Q}, \partial(\mathcal{T}\times\mathcal{M})\big) > 0$, *there is a constant* $C = C\left(n, \lambda, k, l, \alpha, \delta, \|a\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M},\operatorname{Sym}_2\mathbb{R}^n)}, \|b\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M})}\right) < \infty$ *such that*

$$\|u\|_{\mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}(\mathcal{Q})} \leq C\|u\|_{\mathrm{L}^\infty(\mathcal{T}\times\mathcal{M})}. \tag{B.9}$$

Moreover, if $0 \in \mathcal{T}$ *and* $\mathcal{U} \Subset \mathcal{M}$ *with* $\delta_0 := \operatorname{dist}\big(\mathcal{U}, \partial\mathcal{M}\big) > 0$, *there is another constant* $C_0 = C_0\left(n, \lambda, k, l, \alpha, \delta_0, \|a\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M},\operatorname{Sym}_2\mathbb{R}^n)}, \|b\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M})}\right) < \infty$ *such that*

$$\|u\|_{\mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}(\mathcal{T}\times\mathcal{U})} \leq C\left(\|u(0,\cdot)\|_{\mathrm{C}^{(2l+2)+\alpha}(\mathcal{M})} + \|u\|_{\mathrm{L}^\infty(\mathcal{T}\times\mathcal{M})}\right). \tag{B.10}$$

If we consider a solution $u \in \mathrm{C}^{k,l}(\mathcal{T}\times\mathcal{M})$ of the quasi-linear equation

$$\frac{\partial}{\partial t}u(t,p) = \Big\langle A\big(t,p,u(t,p),\mathrm{D}u(t,p)\big), \operatorname{Hess} u(t,p)\Big\rangle + B\big(t,p,u(t,p),\mathrm{D}u(t,p)\big), \tag{B.11}$$

we may define $a \in \mathrm{C}^{k,l-1}\big(\mathcal{T}\times\mathcal{M}, \operatorname{Sym}_2\mathbb{R}^n\big)$ and $b \in \mathrm{C}^{k,l-1}(\mathcal{T}\times\mathcal{M})$ by $a(t,p) := A\big(t,p;u(t,p),\mathrm{D}u(t,p)\big)$ and $b(t,p) := B\big(t,p;u(t,p),\mathrm{D}u(t,p)\big)$ and obtain a linear equation (B.7). This way one can exploit the parabolic Schauder estimates for linear equations also in the quasi-linear context. Note that if $A(t,p;w,\boldsymbol{\xi})$ or $B(t,p;w,\boldsymbol{\xi})$ does not depend on $\boldsymbol{\xi}$ then the associated $a(t,p)$ or $b(t,p)$ respectively is of the same regularity as u.

Corollary B.2.2. *For some $\alpha \in (0,1]$, $k,l \in \mathbb{N}_0$, some time interval $\mathcal{T} \subset \mathbb{R}$ and domain $\mathcal{M} \Subset \mathbb{R}^n$ let $u \in \mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}\big(\mathcal{T} \times \mathcal{M}\big)$ be a solution of (B.11) such that $A(t,p;w,\boldsymbol{\xi}) = A(t,p;w)$ and $B(t,p;w,\boldsymbol{\xi}) = B(t,p;w)$ are independent of $\boldsymbol{\xi}$ and there is some $\lambda > 0$ such that for all $(t,p) \in \mathcal{T} \times \mathcal{M}$*

$$0 < \lambda|\boldsymbol{\eta}|^2 \leq \Big\langle A\big(t,p;u(t,p)\big), \boldsymbol{\eta} \otimes \boldsymbol{\eta} \Big\rangle \qquad \text{for all } \boldsymbol{\eta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{B.12}$$

For any $\mathcal{Q} \Subset \mathcal{T} \times \mathcal{M}$ with $\delta := \operatorname{dist}\big(\mathcal{U}, \partial(\mathcal{T} \times \mathcal{M})\big) > 0$, there is a constant $C = C\left(n, \lambda, k, l, \alpha, \delta, \|u\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M})}\right) < \infty$ such that

$$\|u\|_{\mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}(\mathcal{Q})} \leq C\|u\|_{L^\infty(\mathcal{T}\times\mathcal{M})}. \tag{B.13}$$

Moreover, if $0 \in \mathcal{T}$ and $\mathcal{U} \Subset \mathcal{M}$ with $\delta_0 := \operatorname{dist}\big(\mathcal{U}, \partial\mathcal{M}\big) > 0$, there is another constant $C = C\left(n, \lambda, k, l, \alpha, \delta_0, \|u\|_{\mathrm{C}^{k+\alpha/2,2l+\alpha}(\mathcal{T}\times\mathcal{M})}\right) < \infty$ such that

$$\|u\|_{\mathrm{C}^{(k+1)+\alpha/2,(2l+2)+\alpha}(\mathcal{T}\times\mathcal{U})} \leq C\left(\|u(0,\cdot)\|_{\mathrm{C}^{(2l+2)+\alpha}(\mathcal{M})} + \|u\|_{\mathrm{L}^\infty(\mathcal{T}\times\mathcal{M})}\right). \tag{B.14}$$

B.3 Auxiliary results

In order to bootstrap C^0 convergence into C^k convergence (using C^l bounds) we will need to be able to interpolate:

Lemma B.3.1. *Let $u : \mathbb{B}^n \to [-1,1]$ be a smooth function such that for all $k \in \mathbb{N}$*

$$\|\mathrm{D}^k u\|_{\mathrm{L}^\infty(\mathbb{B})} < \infty.$$

Then for all $k \in \mathbb{N}$ and $\eta \in (0,1)$ there exist constants $C = C(k,\eta) > 0$ and $l := \lceil k/\eta \rceil$ such that

$$|\mathrm{D}^k u|(0) \leq C\left(1 + \|\mathrm{D}^l u\|_{\mathrm{L}^\infty(\mathbb{B})}\right)\|u\|_{\mathrm{L}^\infty(\mathbb{B})}^{1-\eta}. \tag{B.15}$$

Proof. By [GT98, Theorem 7.28] (for example) for arbitrary $0 < k < l$ there is a constant $C = C(l) > 0$ such that for any smooth function $v \in \mathrm{C}^\infty(\mathbb{B}) \cap \mathrm{L}^\infty(\mathbb{B})$ with bounded derivatives

$$\|\mathrm{D}^k v\|_{\mathrm{L}^\infty(\mathbb{B})} \leq C\left(\|v\|_{\mathrm{L}^\infty(\mathbb{B})} + \|\mathrm{D}^l v\|_{\mathrm{L}^\infty(\mathbb{B})}\right). \tag{B.16}$$

Now define $v(x) := u(\varepsilon x)$ for all $x \in \mathbb{B}$ with $\varepsilon = \|u\|_{\mathrm{L}^\infty(\mathbb{B})}^{\frac{1}{l}} \in (0,1]$ and choosing $l = \lceil k/\eta \rceil$ we estimate with (B.16)

$$\begin{aligned}
\|\mathrm{D}^k u\|_{\mathrm{L}^\infty(\mathbb{B}_\varepsilon)} = \varepsilon^{-k}\|\mathrm{D}^k v\|_{\mathrm{L}^\infty(\mathbb{B})} &\leq C\varepsilon^{-k}\left(\|v\|_{\mathrm{L}^\infty(\mathbb{B})} + \|\mathrm{D}^l v\|_{\mathrm{L}^\infty(\mathbb{B})}\right) \\
&\leq C\left(\varepsilon^{-k}\|u\|_{\mathrm{L}^\infty(\mathbb{B}_\varepsilon)} + \varepsilon^{l-k}\|\mathrm{D}^l u\|_{\mathrm{L}^\infty(\mathbb{B}_\varepsilon)}\right) \\
&\leq C\left(1 + \|\mathrm{D}^l u\|_{\mathrm{L}^\infty(\mathbb{B})}\right)\|u\|_{\mathrm{L}^\infty(\mathbb{B})}^{1-\frac{k}{l}} \\
&\leq C\left(1 + \|\mathrm{D}^l u\|_{\mathrm{L}^\infty(\mathbb{B})}\right)\|u\|_{\mathrm{L}^\infty(\mathbb{B})}^{1-\eta}.
\end{aligned}$$

□

Occasionally we will have to switch between equivalent metrics in arguments, and will use the following elementary fact:

Lemma B.3.2. *Let $(\mathbb{D}, g_{\mathbb{H}})$ be the complete hyperbolic disc and $T \in \Gamma\left(\mathbb{D}; \mathrm{T}^{(r,s)}\mathbb{D}\right)$ any (r,s) tensor field. Then for every $k \in \mathbb{N}_0$ and $\varrho \in (0,1)$ there exists a constant $C = C(k, \varrho, r, s) > 0$ such that*

$$\frac{1}{C}\|T\|_{\mathrm{C}^k(\mathbb{D}_\varrho, |\mathrm{d}z|^2)} \leq \|T\|_{\mathrm{C}^k(\mathbb{D}_\varrho, g_{\mathbb{H}})} \leq C\|T\|_{\mathrm{C}^k(\mathbb{D}_\varrho, |\mathrm{d}z|^2)}. \tag{B.17}$$

In particular, we have

$$\frac{1}{C}\sum_{j=0}^{k}\left|\left({}^{|\mathrm{d}z|^2}\nabla\right)^j T\right|_{|\mathrm{d}z|^2}(0) \leq \sum_{j=0}^{k}\left|\left({}^{g_{\mathbb{H}}}\nabla\right)^j T\right|_{g_{\mathbb{H}}}(0) \leq C\sum_{j=0}^{k}\left|\left({}^{|\mathrm{d}z|^2}\nabla\right)^j T\right|_{|\mathrm{d}z|^2}(0).$$

Finally, the following lemma clarifies the differential of the $[\,\cdot\,]_+$-function.

Lemma B.3.3. *For some $T > 0$ and $\mathcal{U} \subset \mathbb{R}^n$ consider the weakly differentiable function $u \in \mathrm{W}^1\big([0,T] \times \mathcal{U}\big)$. Then $t \mapsto \int_{\mathcal{U}}[u(t,p)]_+ \,\mathrm{d}\mu$ is weakly differentiable and we have*

$$\left.\frac{\partial}{\partial t}\right|_{t_0}\int_{\mathcal{U}}\big[u(t,p)\big]_+ \,\mathrm{d}\mu = \int_{\mathcal{U}}\Theta\big(u(t,p)\big)\left.\frac{\partial}{\partial t}\right|_{t_0} u(t,p)\,\mathrm{d}\mu \quad \textit{for a.e. } t_0 \in (0,T), \tag{B.18}$$

where $\Theta = \mathbf{1}_{(0,\infty)} : \mathbb{R} \to \{0,1\}$ is the Heaviside step function.

Proof. By [GT98, Lemma 7.6], $\big[u(t,p)\big]_+$ is weakly differentiable and we have $\mathrm{D}_t[u]_+(t,p) = \Theta\big(u(t,p)\big)\,\mathrm{D}_t u(t,p)$ for a.e. $(t,p) \in (0,T) \times \mathcal{U}$. For some test function $\phi \in C_c^\infty\big((0,T)\big)$ we calculate

$$\begin{aligned}
\int_0^T\left(\mathrm{D}_t\int_{\mathcal{U}}\big[u(t,p)\big]_+\,\mathrm{d}\mu\right)\phi\,\mathrm{d}t &= -\int_0^T\int_{\mathcal{U}}\big[u(t,p)\big]_+\,\mathrm{d}\mu\,\mathrm{D}_t\phi\,\mathrm{d}t \\
&= -\int_{\mathcal{U}}\int_0^T\big[u(t,p)\big]_+\mathrm{D}_t\phi\;\mathrm{d}t\;\mathrm{d}\mu \\
&= \int_{\mathcal{U}}\int_0^T\Theta\big(u(t,p)\big)\,\mathrm{D}_t u(t,p)\,\phi\;\mathrm{d}t\;\mathrm{d}\mu \\
&= \int_0^T\left(\int_{\mathcal{U}}\Theta\big(u(t,p)\big)\,\mathrm{D}_t u(t,p)\;\mathrm{d}\mu\right)\phi\,\mathrm{d}t. \qquad \square
\end{aligned}$$

Bibliography

[AAR09] Albin, Pierre, Clara L. Aldana and Frédéric Rochon: *Ricci flow and the determinant of the Laplacian on non-compact surfaces*. arXiv:0909.0807v2, 29 October 2009.

[BBD88] Burelbach, James P., Seymour G. Bankoff and Stephen H. Davis: *Nonlinear stability of evaporating/condensing liquid films*. Journal of Fluid Mechanics, **195**:463–494, 1988.

[Bol41] Bol, Gerrit: *Isoperimetrische Ungleichungen für Bereiche auf Flächen*. Jahresbericht der Deutschen Mathematiker-Vereinigung, **51**:219–257, 1941.

[BS09] Brendle, Simon and Richard Schoen: *Manifolds with 1/4-pinched curvature are space forms*. Journal of the American Mathematical Society, **22**(1):287–307, January 2009.

[BSY94] Bartz, J., Michael Struwe and Rugang Ye: *A new approach to the Ricci flow on* S^2. Annali della Scuola Normale Superiore di Pisa – Classe di Scienze. 4^e série, **21**(3):475–482, 1994.

[CCG$^+$08] Chow, Bennet, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo and Lei Ni: *The Ricci Flow: Techniques and Applications: Part II: Analytic Aspects*, volume 144 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2008.

[Cha06] Chavel, Isaac: *Riemannian geometry: a modern introduction*. Number 98 in *Cambridge studies in advanced mathematics*. Cambridge University Press, 2. edition, 2006.

[Che01] Chen, X. X.: *Calabi flow in Riemann surfaces revisited: a new point of view*. International Mathematics Research Notices, **2001**(1):275–297, 2001.

[Che09] Chen, Bing-Long: *Strong uniqueness of the Ricci flow*. Journal of Differential Geometry, **82**(2):363–382, 2009.

[Cho91] Chow, Bennet: *The Ricci flow on the 2-sphere*. Journal of Differential Geometry, **33**(2):325–334, 1991.

[Cho97] Chow, Bennet: *Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations*. Communications in Analysis and Geometry, **5**(2):389–409, 1997.

[CK04] Chow, Bennet and Dan Knopf: *The Ricci Flow: An Introduction*, volume 110 of *Mathematical surveys and monographs*. AMS, 2004.

[CLN06] Chow, Bennett, Peng Lu and Li Ni: *Hamilton's Ricci Flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, 2006.

[CLT06] Chen, Xiuxiong, Peng Lu and Gang Tian: *A note on uniformization of Riemann surfaces by Ricci flow*. Proc. Amer. Math. Soc., **134**(11):3391–3393, November 2006.

[CRW11] Cabezas-Rivas, Esther and Burkhard Wilking: *How to produce a Ricci Flow via Cheeger-Gromoll exhaustion*. arXiv:1107.0606, July 2011.

[CZ06] Chen, Bing-Long and Xi-Ping Zhu: *Uniqueness of the Ricci flow on complete noncompact manifolds*. Journal of Differential Geometry, **74**(1):119–154, 2006.

[DD96] DiBenedetto, Emmanuele and David J. Diller: *About a Singular Parabolic Equations Arising in Thin Film Dynamics and in the Ricci Flow for Complete* $\mathbb{R}^2$. In Marcellini, Paolo, Giorgio G. Talenti and Edoardo Vesentini (editors): *Partial differential equations and applications: collected papers in honor of Carlo Pucci*, volume 177 of *Lecture notes in pure and applied mathematics*, pages 103–119. CRC Press, 1996.

[DDD96] Davis, Stephen H., Emmanuele DiBenedetto and David J. Diller: *Some a Priori Estimates for a Singular Evolution Equation Arising in Thin-Film Dynamics*. SIAM Journal on Mathematical Analysis, **27**(3):638–660, 1996.

[DdP95] Daskalopoulos, Panagiota and Manuel A. del Pino: *On a singular diffusion equation*. Communications in Analysis and Geometry, **3**(3):523–542, 1995.

[DdP07] Daskalopoulos, Panagiota and Manuel del Pino: *Type II collapsing of maximal solutions to the Ricci flow in* $\mathbb{R}^2$. Annales de l'institut Henri Poincaré (C) Analyse non linéaire, **24**(6):851–874, 2007.

[DeT03] DeTurck, Dennis M.: *Deforming metrics in the direction of their Ricci tensors (Improved version)*. In Cao, Huai-Dong, Bennett Chow, Sun-Chin Chu and Shing-Tung Yau (editors): *Collected Papers on Ricci Flow*, number 37 in *Series in Geometry and Topology*, chapter 3, pages 163–165. International Press, Somerville, Massachusetts, 2003.

[DH04] Daskalopoulos, Panagiota and Richard S. Hamilton: *Geometric Estimates for the Logarithmic Fast Diffusion Equation*. Communications in Analysis and Geometry, **12**(1):143–164, 2004.

[DS10] Daskalopoulos, Panagiota and Natasa Sesum: *Type II Extinction Profile of Maximal Solutions to the Ricci Flow in* $\mathbb{R}^2$. Journal of Geometric Analysis, **20**(3):565–591, 2010.

[ES64] Eells, James and Joseph Harold Sampson: *Harmonic Mappings of Riemannian Manifolds*. American Journal of Mathematics, **86**(1):109–160, January 1964.

[Gie07] Giesen, Gregor: *Der Ricci-Fluß auf geschlossenen Flächen und Uniformisierung*. Diploma thesis, Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, 72076 Tübingen, Germany, July 2007.

[GNN79] Gidas, Basilis, Wei-Ming Ni and Louis Nirenberg: *Symmetry and related properties via the maximum principle*. Communications in Mathematical Physics, **68**(3):209–243, 1979.

[GT98] Gilbarg, David and Neil S. Trudinger: *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, revised 3rd printing of the 2nd edition, 1998.

[GT10] Giesen, Gregor and Peter M. Topping: *Ricci flow of negatively curved incomplete surfaces*. Calculus of Variations and Partial Differential Equations, **38**(3):357–367, July 2010.

[GT11] Giesen, Gregor and Peter M. Topping: *Existence of Ricci Flows of Incomplete Surfaces*. Communications in Partial Differential Equations, **36**(10):1860–1880, 2011.

[GT12] Giesen, Gregor and Peter M. Topping: *Ricci flows with unbounded curvature*. Mathematische Zeitschrift, 2012. to appear.

[Ham82] Hamilton, Richard S.: *Three-manifolds with positive Ricci curvature*. Journal of Differential Geometry, **17**(2):255–306, 1982.

[Ham88] Hamilton, Richard S.: *The Ricci flow on surfaces*. In Isenberg, James A. (editor): *Mathematics and general relativity (Santa Cruz, CA, 1986)*, volume 71 of *Contemporary Mathematics*, pages 237–262, Providence, RI, 1988. American Mathematical Society.

[Ham95a] Hamilton, Richard S.: *A Compactness Property for Solutions of the Ricci Flow*. American Journal of Mathematics, **117**(3):545–572, June 1995.

[Ham95b] Hamilton, Richard S.: *The formation of singularities in the Ricci flow*. In *Surveys in differential geometry (Cambridge, MA, 1993)*, volume II, pages 7–136, Cambridge, MA, 1995. International Press.

[Ham95c] Hamilton, Richard S.: *An isoperimetric estimate for the Ricci flow on the two-sphere*. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, number 137 in *Annals of Mathematics Studies*, pages 191–200, Princeton, NJ, 1995. Princeton Univ. Press.

[HP58] Helfferich, Friedrich and Milton S. Plesset: *Ion Exchange Kinetics. A Nonlinear Diffusion Problem*. Journal of Chemical Physics, **28**(3):418–424, March 1958.

[Hub57] Huber, Alfred: *On subharmonic functions and differential geometry in the large*. Commentarii Mathematici Helvetici, **32**:13–72, 1957.

[Hui02] Hui, King Ming: *On some Dirichlet and Cauchy problems for a singular diffusion equation*. Differential and Integral Equations, **15**(7):769–804, July 2002.

[IJ09] Isenberg, James and Mohammad Javaheri: *Convergence of Ricci Flow on $\mathbb{R}^2$ to Flat Space*. Journal of Geometric Analysis, **19**(4):809–816, October 2009.

[IMS11] Isenberg, James, Rafe Mazzeo and Natasa Sesum: *Ricci Flow in Two Dimensions*. To appear in the volume “Surveys in Geometric Analysis and Relativity celebrating Richard Schoen’s 60th birthday.” `arXiv:1103.4669v1`, 2011.

[IMS12] Isenberg, James, Rafe Mazzeo and Natasa Sesum: *Ricci flow on asymptotically conical surfaces with nontrivial topology*. Journal für die reine und angewandte Mathematik, 2012. `arXiv:1003.5237v1`.

[JMS09] Ji, Lizhen, Rafe Mazzeo and Natasa Sesum: *Ricci flow on surfaces with cusps*. Mathematische Annalen, **345**(4):819–834, December 2009.

[Kin93] King, John R.: *Self-Similar Behaviour for the Equation of Fast Nonlinear Diffusion*. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, **343**(1668):337–375, 15 May 1993.

[Koe07] Koebe, Paul: *Über die Uniformisierung beliebiger analytischer Kurven*. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, pages 197–210, 1907.

[Lee00] Lee, John M.: *Introduction to Topological Manifolds*, volume 202 of *Graduate texts in mathematics*. Springer-Verlag, 2000.

[LH76] Lonngren, Karl E. and Akira Hirose: *Expansion of an electron cloud*. Physics Letters A, **59**(4):285–286, 13 December 1976.

[LSU68] Ladyženskaja, Olga Aleksandrovna, Vsevolod Alekseevich Solonnikov and Nina Nikolaevna Ural’ceva: *Linear and Quasi-linear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, 1968.

[McK75] McKean, Henry P.: *The central limit theorem for Carleman’s equation*. Israel Journal of Mathematics, **21**(1):54–92, 1975.

[Omo67] Omori, Hideki: *Isometric immersions of Riemannian manifolds*. Journal of the Mathematical Society of Japan, **19**(2):205–214, 1967.

[Per02] Perelman, Grisha: *The entropy formula for the Ricci flow and its geometric applications*. `arXiv:math/0211159`, 11 November 2002.

[Per03a] Perelman, Grisha: *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*. `arXiv:math/0307245`, 17 July 2003.

[Per03b] Perelman, Grisha: *Ricci flow with surgery on three-manifolds*. `arXiv:math/0303109`, 10 March 2003.

[Poi07] Poincaré, Henri: *Sur l’uniformisation des fonctions analytiques*. Acta Mathematica, **31**(1):1–63, 1907.

[RV95] Rodriguez, Ana and Juan Luis Vazquez: *Non-uniqueness of solutions of nonlinear heat equations of fast diffusion type*. Annales de l’institut Henri Poincaré (C) Analyse non linéaire, **12**(2):173–200, 1995.

[RVE97] RODRIGUEZ, ANA, JUAN LUIS VAZQUEZ and JUAN RAMÓN ESTEBAN: *The maximal solution of the logarithmic fast diffusion equation in two space dimensions.* Advancees in Differential Equations, **2**(6):867–894, November 1997.

[Shi89] SHI, WAN-XIONG: *Deforming the metric on complete Riemannian manifolds*. Journal of Differential Geometry, **30**(1):223–301, 1989.

[SSS08] SCHNÜRER, OLIVER C., FELIX SCHULZE and MILES SIMON: *Stability of Euclidean space under Ricci flow.* Communications in Analysis and Geometry, **16**(1):127–158, January 2008.

[SSS11] SCHNÜRER, OLIVER C., FELIX SCHULZE and MILES SIMON: *Stability of hyperbolic space under Ricci flow.* Communications in Analysis and Geometry, **19**(5):1023–1047, December 2011.

[SST03] SHIOHAMA, KATSUHIRO, TAKASHI SHIOYA and MINORU TANAKA: *The Geometry of Total Curvature on Complete Open Surfaces.* Number 159 in *Cambridge tracts in mathematics.* Cambridge University Press, 2003.

[Str02] STRUWE, MICHAEL: *Curvature flows on surfaces.* Annali della Scuola Normale Superiore di Pisa – Classe di Scienze. 5^{e} série, **1**(2):247–274, 2002.

[Thu82] THURSTON, WILLIAM P.: *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*. Bulletin of the American Mathematical Society, **6**:357–381, 1982.

[Top98] TOPPING, PETER: *Mean curvature flow and geometric inequalities*. Journal für die reine und angewandte Mathematik, **503**:47–61, 1998.

[Top99] TOPPING, PETER: *The isoperimetric inequality on a surface*. manuscripta mathematica, **100**(1):23–23, 1999.

[Top05] TOPPING, PETER M.: *Notes on Pseudolocality.* 2005.

[Top06] TOPPING, PETER: *Lectures on the Ricci Flow*. Number 325 in *London Mathematical Society Lecture Note Series.* Cambridge University Press, October 2006. `http://www.warwick.ac.uk/~maseq/RFnotes.html`.

[Top10] TOPPING, PETER: *Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics*. Journal of the European Mathematical Society, **12**(6):1429–1451, 2010.

[Top11] TOPPING, PETER M.: *Remarks on Hamilton's Compactness Theorem for Ricci flow*. arXiv:1110.3714, October 2011.

[Top12] TOPPING, PETER M.: *Uniqueness and Nonuniqueness for Ricci Flow on Surfaces: Reverse Cusp Singularities*. International Mathematics Research Notices, **2012**(10):2356–2376, 2012.

[Vaz92] VAZQUEZ, JUAN LUIS: *Nonexistence of Solutions for Nonlinear Heat Equations of Fast-Diffusion Type*. Journal de mathématiques pures et appliquées, **71**(6):503–526, 1992.

[WD82] Williams, Malcolm B. and Stephen H. Davis: *Nonlinear theory of film rupture*. Journal of Colloid and Interface Science, **90**(1):220–228, November 1982.

[Wu93] Wu, Lang-Fang: *The Ricci flow on complete* $\mathbb{R}^2$. Communications in Analysis and Geometry, **1**(3):439–472, 1993.

[Yau73] Yau, Shing-Tung: *Remarks on conformal transformations*. Journal of Differential Geometry, **8**(3):369–381, 1973.

[Yau75] Yau, Shing-Tung: *Harmonic functions on complete riemannian manifolds*. Communications on Pure and Applied Mathematics, **28**(2):201–228, 1975.

[Ye94] Ye, Rugang: *Global existence and convergence of Yamabe flow*. Journal of Differential Geometry, **39**(1):35–50, 1994.